

Research Problem

Elementary Divisors of an Inversion-Counting Matrix

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Let $A_n(q)$ denote the $n!$ -square matrix (with rows and columns indexed by the full symmetric group S_n) whose (σ, τ) entry is $q^{\text{Inv}(\sigma\tau^{-1})}$ where q is an indeterminate and $\text{Inv}(\pi)$ denotes the number of inversions in π . This matrix has found application to the q -permanent [1] and to a model in “infinite statistics” [2]. Several authors [2] [3] [4] prove that $A_n(q)$ is positive definite for $-1 < q < 1$. D. Zagier [2] computes the determinant

$$\det A_n(q) = \prod_{k=1}^{n-1} (1 - q^{k(k+1)})^{\frac{n!(n-k)}{k(k+1)}}$$

by factoring the element $\sum_{\pi \in S_n} q^{\text{Inv}(\pi)} \pi$ in the group ring $(\mathbb{Z}[q])[S_n]$, and conjectures that the entries of $[\prod_{k=1}^{n-1} (1 - q^{k(k+1)})] A_n(q)^{-1}$ are polynomials in q with integral coefficients. We strengthen Zagier’s conjecture slightly.

CONJECTURE 1 Suppose the entries of $A_n(q)^{-1}$ are written in lowest terms as quotients of integer-coefficient polynomials. Then the least common multiple (lcm) of the denominators $\stackrel{?}{=} \prod_{k=1}^{n-1} (q^{k(k+1)} - 1)$.

The determinantal divisors $(d_i(q) := \gcd\{i \times i \text{ minors}\})$ of $A_n(q)$ are monic integral polynomials, hence so are its invariant factors. $\psi_i(q) := d_i(q)/d_{i-1}(q)$. Recall $\psi_i(q)$ divides $\psi_{i+1}(q)$, $1 \leq i \leq n! - 1$, and $\prod_{i=1}^{n!} \psi_i(q) = \det A_n(q)$. Conjecture 1 is equivalent to

CONJECTURE 2 $\psi_{n!}(q) \stackrel{?}{=} \prod_{k=1}^{n-1} (q^{k(k+1)} - 1)$.

To see this equivalence, write A for $A_n(q)$ for brevity, and recall that from the definitions, the gcd of the cofactors of A is $d_{n!-1} = \prod_{i=1}^{n!-1} \psi_i = \det A / \psi_{n!}$, equivalently, the gcd of the entries of $(\det A) A^{-1}$ is $\det A / \psi_{n!}$. Hence for the entries of A^{-1} (in lowest terms), the gcd of the numerators is 1, and the lcm of the denominators is $\psi_{n!}$. $\square$

This raises the question of describing all the invariant factors of $A_n(q)$ or equivalently, all its elementary divisors (prime power divisors of its invariant

factors). For example, the unimodular matrices P, Q below serve to put $A_3(q)$ in Smith normal form (invariant factors on the diagonal): $PA_3Q = \text{diag}(1, 1 - q^2, 1 - q^2, 1 - q^2, (1 - q^2)^2, (1 - q^2)(1 - q^6))$.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 & 0 & 0 \\ -q & 0 & 1 & 0 & 0 & 0 \\ 0 & -q & 0 & 1 & 0 & 0 \\ 0 & q^3 & -q & -q^2 & 1 & 0 \\ q^3 & q^6 & -q^2 - q^4 & -q - q^5 & q^3 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & -q & -q & 0 & -q^4 & q^3 + q^5 \\ 0 & 1 & 0 & -q & q^3 & -q^4 \\ 0 & 0 & 1 & 0 & -q + q^3 & -q^4 \\ 0 & 0 & 0 & 1 & 0 & -q \\ 0 & 0 & 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 0 & -q & 1 + q^2 \end{pmatrix}$$

Clearly, from the factorization of $\det A_n(q)$ above, every elementary divisor of $A_n(q)$ (over $\mathbb{Q}$) is a power of one of the cyclotomic polynomials $\phi_1(q) = q - 1$, $\phi_2(q) = q + 1$, $\phi_3(q) = q^2 + q + 1$, $\dots$. Since $(A_n(-q))_{\sigma\tau} = (-q)^{\text{Inv}(\sigma\tau^{-1})} = \text{sgn } \sigma \text{sgn } \tau (A_n(q))_{\sigma\tau}$, it follows that all invariant factors of $A_n(q)$ are polynomials in q^2 . Also, $\phi_{4k}(-q) = \phi_{4k}(q)$, while $\phi_{2k+1}(-q) = \phi_{2(2k+1)}(q)$. Together with the irreducibility of the cyclotomic polynomials, these facts imply ϕ_{2k+1}^j and $\phi_{2(2k+1)}^j$ always have the same multiplicity as elementary divisors of $A_n(q)$. The following table gives the elementary divisors of $A_3(q)$ and $A_4(q)$ [multiplicities in brackets]. (The entries for $A_4(q)$ were obtained using *Mathematica*.)

$A_3(q)$		$A_4(q)$	
$\phi_1[3]$	$\phi_1^2[2]$	$\phi_1[6]$	$\phi_1^2[11] \quad \phi_1^3[6]$
$\phi_2[3]$	$\phi_2^2[2]$	$\phi_2[6]$	$\phi_2^2[11] \quad \phi_2^3[6]$
$\phi_3[1]$		$\phi_3[6]$	$\phi_3^2[2]$
$\phi_6[1]$		$\phi_4[2]$	
		$\phi_6[6]$	$\phi_6^2[2]$
		$\phi_{12}[2]$	

The multiplicities for ϕ_1 and ϕ_2 seem to involve the Stirling cycle numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ — the number of permutations in S_n with exactly k cycles — also known as the unsigned Stirling numbers of the first kind.

CONJECTURE 3 *The list of elementary divisors of $A_n(q)$ contains exactly $\left[\begin{smallmatrix} n \\ n-i \end{smallmatrix} \right]$ copies of $(q-1)^i$ and $\left[\begin{smallmatrix} n \\ n-i \end{smallmatrix} \right]$ copies of $(q+1)^i$ for each i .*

Conjecture 3 jibes with Conjecture 2 in that both imply $n-1$ is the largest power of $q-1$ (resp. $q+1$) that divides $\psi_n(q)$, and it also yields the correct total power of $(1-q^2)$ that divides $\det A_n(q)$ as follows. From the factorization of $\det A_n(q)$ above, this power is $\sum_{k=1}^n \frac{n!(n-k)}{k(k+1)} = n!(n-1-\frac{1}{2}-\dots-\frac{1}{n})$, by elementary telescoping sums. On the other hand, according to Conjecture 3, this power is $\sum_{i=0}^n \begin{bmatrix} n \\ n-i \end{bmatrix} i = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} (n-i) = n! \cdot n - \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} i = n!(n-1-\frac{1}{2}-\dots-\frac{1}{n})$, using an elementary Stirling number identity.

The invariant factors for $A_4(q)$ are [multiplicities in brackets]: $1[1], q^2-1[6], (q^2-1)^2[9], (q^2-1)(q^6-1)[2], (q^2-1)^2(q^6-1)[4], (q^2-1)(q^6-1)(q^{12}-1)[2]$. This suggests that possibly the following might be true.

CONJECTURE 4 *Each invariant factor of $A_n(q)$ is a product of powers of polynomials from the set $\{(q^2-1)^{k(k+1)}\}_{1 \leq k \leq n-1}$.*



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References

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