



Another Way to Discover That $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

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EXAMPLE 3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $AB = I$ and it is trivial to check that $Ax = 0$ implies $x = 0$, so B is the unique right inverse of A . To see that A has no left inverse matrix, suppose r is the first row of such a matrix. Then $\sum r_i = 1$ but $\sum_{i=1}^n r_i = 0$ for $n = 1, 2, \dots$. This is impossible.

The matrix A in EXAMPLE 3, considered as a matrix map from cs to c , has an inverse map which is not a matrix map. To see what the inverse map M is, let $x \in c$, and $y = M(x) \in cs$. Then $x = Ay$ so that $x_n = y_1 + \sum_{k=n+1}^{\infty} y_k$ from which $y_1 = \lim x_n$, $y_{n+1} = x_n - x_{n+1}$, i.e., $y = T(x) + Bx$, where T is defined in EXAMPLE 1 and B is defined in EXAMPLE 3. Thus $M = T + B$, which (using an argument as in EXAMPLE 1) is not a matrix map. The matrix A in EXAMPLE 3 does not have an inverse matrix.

An example is given in [3], p. 407, Theorem 11, of a matrix which has an inverse matrix and an inverse map which are different, i.e., the inverse map is not given by the inverse matrix! This illustrates a curious possibility for our Theorem: let T be the map given by the matrix A such that T^{-1} is not given by A^{-1} ; $T \circ T^{-1}$ is given by the matrix $AA^{-1} = I$.

In [1], pp. 245–247, are shown several symmetric row-finite matrices each of which has infinitely many two-sided inverse matrices. The inverses shown are also symmetric. Let A be such a matrix, i.e., A is symmetric, row-finite, has a symmetric two-sided inverse and a symmetric nonzero matrix Z such that $ZA = AZ = 0$ (Z is the difference between two inverses.) It is proved in [2], Theorem 3c, that A has a row-finite right inverse R . Paradoxically R cannot be symmetric for if it were we would have $RA = I$ and so $0 = R(AZ) = (RA)Z = Z$.

A talk given to the Eastern Pennsylvania and Delaware Section of the MAA on November 19, 1983.

References

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Another Way to Discover that

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n2^{n+1}n!} = \ln 2$$

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The article [1] caught my eye. That identity looks familiar, I thought, and then I realized that I had recently discovered the same identity by a method which is also somewhat serendipitous. This involved consideration of the following problem in probability.

The World Series is played as the best of seven games. Suppose that the better team (say A) has a fixed probability p ($> 1/2$) of winning any particular game and that, more generally, a series of $2n+1$ games is played. What is the probability P_n that A wins the series?

First, it is intuitively plausible that if enough games are played, the better team is almost sure to come out on top, i.e., $\lim_{n \rightarrow \infty} P_n = 1$. This can be established using Tchebychev's well-known

inequality [2]. Let X denote the number of games A wins. (Here it is convenient and harmless to suppose that all $2n+1$ games are played, although in practice, of course, the series would be stopped as soon as one team reached $n+1$ wins.) Then X has the binomial distribution: $b(2n+1, p)$. Now $P_n = \Pr(X \geq n+1) = 1 - \Pr(X \leq n)$, and we find that (recall that the mean of X is $(2n+1)p$)

$$\begin{aligned}\Pr(X \leq n) &\leq \Pr(|X - (2n+1)p| \geq (2n+1)p - n) \\ &\leq \frac{\text{Var } X}{((2n+1)p - n)^2} \quad (\text{by Tchebychev}) \\ &= \frac{npq}{n^2 \left(2p + \frac{1}{n}p - 1\right)^2} \\ &\leq \frac{pq}{n(2p-1)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

This establishes that $\lim_{n \rightarrow \infty} P_n = 1$. Next let us examine the individual P_n . We observe that $P_n = \Pr(X \geq n+1)$ is a sum of binomial probabilities (with $q \equiv 1-p$):

$$P_n = p^{n+1} \left[\binom{n+0}{0} + \binom{n+1}{1} q + \binom{n+2}{2} q^2 + \cdots + \binom{n+n}{n} q^n \right]. \quad (1)$$

Using (1) to compute $P_n - P_{n-1}$ and the basic relation

$$\binom{n+k}{k} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k}$$

to simplify the result, we find

$$P_n - P_{n-1} = p^n q^n \binom{2n-1}{n} (1-2q). \quad (2)$$

(Note this implies that P_n increases with n , which is not surprising. Indeed, it was an effort to establish the positivity of $P_n - P_{n-1}$ which motivated (2).) Summing (2) from 1 to n yields

$$P_n = p + (1-2q) \left(pq + \binom{3}{1} (pq)^2 + \binom{5}{2} (pq)^3 + \cdots + \binom{2n-1}{n} (pq)^n \right). \quad (3)$$

We now take the limit as $n \rightarrow \infty$ of (3), then divide by q and let $x = 4pq$ (which implies $0 \leq x < 1$ and $p = (1 + \sqrt{1-x})/2$) to obtain

$$\frac{2}{\sqrt{1-x} + 1 - x} = 1 + \binom{3}{1} \frac{x}{4} + \binom{5}{2} \frac{x^2}{4^2} + \cdots, \quad 0 \leq x < 1. \quad (4)$$

Integrating (4) term by term (no problem for a series expansion) gives the interesting Maclaurin series (at least for $0 \leq x < 1$)

$$\ln(1 + \sqrt{1-x}) = \ln 2 - \frac{x}{4} - \frac{1}{2} \binom{3}{1} \frac{x^2}{4^2} - \frac{1}{3} \binom{5}{2} \frac{x^3}{4^3} - \cdots. \quad (5)$$

The series on the right in (5) converges at $x=1$ (Raabe's Test), hence converges uniformly on $[-1, 1]$ (Weierstrass M-Test); hence, its sum is continuous on $[-1, 1]$. This permits us to put $x=1$ in (5) to obtain the identity of the title.

References

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