

ASCENT SEQUENCES AVOIDING A TRIPLE OF 3-LETTER PATTERNS AND FIBONACCI NUMBERS

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ABSTRACT. We consider the triples τ of three letter patterns for which ascent sequences that avoid τ are counted by Fibonacci-related numbers. Several cases are equivalent to avoiding a pair of 3-letter patterns and have already been treated using generating functions. Here we give direct proofs of all our results. We also consider avoiders for some quadruples of 3-letter patterns.

1. INTRODUCTION

An *ascent* in an integer sequence $s_1s_2\cdots s_m$ is an index $1 \leq j \leq m-1$ such that $s_j < s_{j+1}$. An *ascent sequence* $a_1a_2\cdots a_n$ is one consisting of non-negative integers satisfying $a_1 = 0$ and for all i with $1 < i \leq n$, $a_i \leq \text{asc}(a_1a_2\cdots a_{i-1}) + 1$, where $\text{asc}(a_1a_2\cdots a_k)$ is the number of ascents in the sequence $a_1a_2\cdots a_k$. For example, the sequence 0101312501 is an ascent sequence, whereas 01003 is not. Bousquet-Mélou, Claesson, Dukes, and Kitaev [2] connected ascent sequences to $(2+2)$ -free posets. Since then, ascent sequences have been studied in a series of papers where connections to many other combinatorial structures have been created (see, for example, [3–5, 7–9] as well as [6, Section 3.2.2]).

Let $a = a_1a_2\cdots a_n$ be an ascent sequence and $\tau = \tau_1\cdots\tau_m$ be any *pattern*, that is, a word in $\{0, \dots, \ell\}^m$ which contains each letter $0, 1, \dots, \ell$. We say that a contains τ if a has a subsequence that is order isomorphic to τ , that is, there is a subsequence $a_{f(1)}, a_{f(2)}, \dots, a_{f(m)}$, where $1 \leq f(1) < f(2) < \cdots < f(m) \leq n$, such that $a_{f(i)}Xa_{f(j)}$ if and only if $\tau_iX\tau_j$, for all $X \in \{<, >, =\}$ and $1 \leq i, j \leq m$. Otherwise, a is said to avoid τ . For instance, the ascent sequence 01013042251 has two occurrences of the pattern 110, namely, the subsequences 110 and 221, but avoids the pattern 3210. We denote the set of all ascent sequences that avoid a list of patterns $\tau^{(1)}, \dots, \tau^{(s)}$ by $A_n(\tau^{(1)}, \dots, \tau^{(s)})$ or $A_n(\{\tau^{(1)}, \dots, \tau^{(s)}\})$.

Mansour and Shattuck [9] determined all pairs of patterns $\{\tau_1, \tau_2\}$ giving rise to Fibonacci numbers where τ_1 and τ_2 are both length-3 patterns or one is length-3 and the other is

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length-4. In particular, they showed that

$$\begin{aligned}
|A_n(000, 001)| &= |A_n(000, 010)| = F_{n+1}, \\
|A_n(021, 101)| &= |A_n(101, 102)| = |A_n(021, 120)| = |A_n(101, 110)| = |A_n(021, 0101)| \\
&= |A_n(021, 0120)| = |A_n(021, 0122)| = |A_n(100, 0121)| = |A_n(101, 0011)| = |A_n(101, 0102)| \\
&= |A_n(101, 0110)| = |A_n(101, 0121)| = |A_n(102, 0001)| = |A_n(102, 0012)| = |A_n(102, 0101)| \\
&= |A_n(110, 0101)| = |A_n(110, 0122)| = |A_n(120, 0123)| = F_{2n-1},
\end{aligned}$$

where F_n denotes the n -th Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$ (sequence A000045 in [10]).

The aim of this paper is to determine all pattern avoidance classes of ascent sequences avoiding a triple of length-3 patterns whose counting sequences are closely related to Fibonacci numbers. More precisely, we show the following result.

Theorem 1. (1) *Let*

$$\begin{aligned}
f_1 &= \{000, 001, 100\}, & f_2 &= \{000, 001, 101\}, & f_3 &= \{000, 001, 102\}, & f_4 &= \{000, 001, 201\}, \\
f_5 &= \{000, 010, 021\}, & f_6 &= \{000, 010, 100\}, & f_7 &= \{000, 010, 101\}, & f_8 &= \{000, 010, 102\}, \\
f_9 &= \{000, 010, 110\}, & f_{10} &= \{000, 010, 120\}, & f_{11} &= \{000, 010, 201\}, & f_{12} &= \{000, 010, 210\}.
\end{aligned}$$

Then $|A_n(f_j)| = F_{n+1}$ for all $j = 1, 2, \dots, 12$.

(2) *Let*

$$g_1 = \{001, 100, 101\}, \quad g_2 = \{001, 100, 102\}, \quad g_3 = \{001, 100, 201\}.$$

Then, $|A_n(g_j)| = F_{n+2} - 1$ for all $j = 1, 2, 3$.

(3) *Let*

$$\begin{aligned}
h_1 &= \{021, 101, 201\}, & h_2 &= \{021, 101, 210\}, & h_3 &= \{021, 120, 201\}, & h_4 &= \{021, 120, 210\}, \\
h_5 &= \{100, 101, 210\}, & h_6 &= \{101, 102, 201\}, & h_7 &= \{101, 110, 201\}, & h_8 &= \{101, 110, 210\}.
\end{aligned}$$

They $|A_n(h_j)| = F_{2n-1}$, for all $j = 1, 2, \dots, 8$.

(4) *Let $i_1 = \{000, 021, 101\}$ and $i_2 = \{000, 021, 110\}$. Then*

$$|A_n(i_1)| = |A_n(i_2)| = F_n + \sum_{k=2}^n F_{k-1} F_{n-k+1}.$$

Additionally, in the last section, we extend our results to some quadruples of 3-letter patterns.

2. PROOF OF THEOREM 1

2.1. Proof of Theorem 1 part (1).

2.1.1. f_1 thru f_4 . Ascent sequences avoiding the pair (000,001) were counted in [1]. We recall their characterization here. For an ascent sequence $w \in A_n(000,001)$, all entries after the first weak descent vu , $v \geq u$, must be $< u$ and strictly decreasing. In particular, for $n \geq 3$, w cannot begin 00 and begins $01 \dots v$ where v is the maximum value. The only ascent sequence in $A_n(000,001)$ with no weak descents is $01 \dots n-1$. Thus, for $n \geq 3$,

$$|A_n(000,001)| = 1 + \sum_{v=1}^{n-2} \sum_{u=0}^v \binom{u}{n-(v+1)} = F_{n+1}.$$

From the preceding characterization, it is clear that $w \in A_n(000,001)$ also avoids 100, 101, 102, and 201. Hence $|A_n(f_1)| = |A_n(f_2)| = |A_n(f_3)| = |A_n(f_4)|$.

2.1.2. f_5 thru f_{12} . Let $\mathcal{F}_n$ denote the set of integer sequences $(u_i)_{i=1}^n$ that satisfy (i) $u_1 = 0$, (ii) $u_i \leq u_{i+1} \leq u_i + 1$ for $1 \leq i \leq n-1$, and (iii) no integer occurs more than twice in the sequence. For example, $\mathcal{F}_4 = \{0011, 0012, 0112, 0122, 0123\}$. It is easy to see that $|\mathcal{F}_n| = F_{n+1}$, where F_n is the Fibonacci number. It is also easy to see that $A_n(000,010)$ coincides with $\mathcal{F}_n$. Since sequences in $\mathcal{F}$ also avoid 021, 100, 101, 102, 110, 120, 201, and 210, all the triples f_5 through f_{12} have the same set of avoiders.

2.2. Proof of Theorem 1 part (2). An ascent sequence $w \in A_n(001,100)$ (see [1] for an alternative account) must begin with unit increments until its maximum entry $m \geq 0$ is achieved, thus $01 \dots m$, and this m then occurs (consecutively) $j \geq 1$ times. Any remaining entries must be $< m$ and strictly decreasing. Consequently,

$$|A_n(001,100)| = \sum_{m=0}^n \sum_{j=1}^{n-m} \binom{m}{n-(m+j)} = F_{n+2} - 1.$$

From the preceding characterization, it is clear that $w \in A_n(001,100)$ also avoids 101, 102 and 201. Hence $|A_n(001,100,101)| = |A_n(001,100,102)| = |A_n(001,100,201)|$.

2.3. Proof of Theorem 1 part (3).

2.3.1. h_1 and h_2 . To suppress consecutive repetition in a sequence, denoted `supp_consec_rep`, means to successively erase from left to right each entry that coincides with its immediate predecessor, in other words, to replace each run of equal entries with a single copy of that entry, thus `supp_consec_rep(0233310044046)=023104046`. Similarly, `del_0` means to erase 0's, thus `del_0(023104046)=231446`.

Proposition 2. *A sequence of n nonnegative integers $w = (w_i)_{i=1}^n$ is in $A_n(021,101)$ if and only if $w_1 = 0$ and `del_0(supp_consec_rep(w))` is empty or an initial segment of the positive integers $1, 2, \dots, m$ for some m . For example, $00111023340055 \in A_{14}(021,101)$ but 0101 and 01221 are not.*

Proof. Easy and left to the reader. Note that the stated condition guarantees w grows slowly enough to be an ascent sequence. $\square$

Let $\mathcal{C}_{m,k}$ denote the set of compositions $(c_i)_{i=1}^k$ of m into k nonnegative parts. The following sums are well known and easy to prove using lattice paths of m East and $k-1$ or $2k-1$ North steps:

$$(2.1) \quad \sum_{c \in \mathcal{C}_{m,k}} 1 = \binom{m+k-1}{m},$$

$$\sum_{c \in \mathcal{C}_{m,k}} \prod_{i=1}^k (c_i + 1) = \binom{m+2k-1}{m}.$$

To form a sequence $w = (w_i)_{i=1}^n$ with j initial 0's and maximum entry k that meets the condition of Proposition 2, note that there are $n-k-j$ entries other than the initial 0's and the first occurrences of $1, 2, \dots, k$. Begin with a composition $(c_i)_{i=1}^k$ of $n-k-j$ where $c_i \geq 0$ is the number of entries strictly between the first occurrence of i and the first occurrence of $i+1$ (or strictly after the first occurrence of k when $i=k$). Each c_i gives c_i+1 choices to determine where the left to right entries switch from i to 0, as they must. All told, the number of (021,101)-avoiding ascent sequences of length n with j initial 0's and maximum entry k is

$$\begin{aligned} 1 \text{ (for the all-0's sequence)} &+ \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \sum_{c \in \mathcal{C}_{n-k-j,k}} \prod_{i=1}^k (c_i + 1) = \\ 1 &+ \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \binom{n+k-j-1}{n-k-j} = \\ 1 &+ \sum_{j=1}^{n-1} F_{2(n-j)} = F_{2n-1}, \end{aligned}$$

using (2.1) for the first equality and standard Fibonacci number identities for the others.

The above characterization shows that (021,101)-avoiders also avoid 201 and 210. Thus h_1 -avoiders and h_2 -avoiders both coincide with (021,101)-avoiders.

2.3.2. h_3 and h_4 . A left-to-right maximum (lrMax) in a sequence w of integers is an entry that is greater than all its predecessors; thus a particular value can occur only once as an lrMax in w , and 002243 has lrMax entries at positions 1,3,5.

Proposition 3. *An ascent sequence $w = (w_i)_{i=1}^n$ avoids 021 and 120 if and only if every descent top in w (if any) is 1. For example, 00100110244455 $\in A_{14}(021,120)$ but 010212 is not.*

Proof. Easy and left to the reader. $\square$

Thus a nonnegative sequence is in $\cup_{n \geq 1} A_n(021,120)$ if and only if it has the form uv where u is an arbitrary sequence of 0's and 1's that starts with a 0 and if v is nonempty then (i) $\max(u) = 1$ so that u has $r \geq 1$ ascents and (ii) the first entry (= first lrMax) of v is ≥ 2

and the i -th lrMax of v is $\leq r + i$, $i \geq 1$. The all-0's sequence u has generating function $1/(1-x)$, where x marks number of entries. Otherwise u has $r \geq 1$ ascents and hence has the form $0^{i_0} \cdot 01 \cdot 1^{i_1} 0^{j_1} \cdot 01 \cdot \dots \cdot 01 \cdot 1^{i_r} 0^{j_r}$ with all exponents ≥ 0 , leading to the generating function $x^{2r}/(1-x)^{2r+1}$. As for v , the “base case” $r = 1$ involves a sequence that starts with a 2 and each successive entry is the same as, or 1 unit greater than, its predecessor leading to the generating function: 1 (for the empty sequence) $+ x/(1-2x) = (1-x)/(1-2x)$. The general case can be seen to be the r -fold convolution of the base case with generating function $((1-x)/(1-2x))^r$. All told, including the empty ascent sequence, the generating function is

$$\frac{1}{1-x} + \sum_{r \geq 1} \frac{x^{2r}}{(1-x)^{2r+1}} \left(\frac{1-x}{1-2x} \right)^r = \frac{1-2x}{1-3x+x^2} = 1 + \sum_{n \geq 1} F_{2n-1} x^n.$$

The above characterization shows that (021,120)-avoiders also avoid 201 and 210. Thus h_3 -avoiders and h_4 -avoiders both coincide with (021,120)-avoiders.

2.3.3. h_5 . To avoid h_5 means that for every inversion $v > u$, all entries after the u are $> u$ and $\neq v$. Now consider a nonempty ascent sequence w . Let $k \geq 1$ denote the position of the last 0 in w and set $m = \max(w_i)_{i=1}^k$. Then it is easy to see that w is a (100,101,210)-avoider if and only if (i) $(w_i)_{i=1}^{k-1}$ is weakly increasing and contains $0, 1, \dots, m$, (ii) $w_{k+1} = m+1$, (iii) $(w_i - m - 1)_{i \geq k+1}$ is a (100,101,210)-avoider. On the other hand, as shown in Section 2.3.5 below, w is a (101,110)-avoider if and only if (i) the nonzero entries of $(w_i)_{i=1}^{k-1}$ are $1, 2, \dots, m$ in that order, (ii) $w_{k+1} = m+1$, (iii) $(w_i - m - 1)_{i \geq k+1}$ is a (101,110)-avoider. Thus a bijection $A_n(101, 110) \rightarrow A_n(100, 101, 210)$ can be defined as follows. Given $w \in A_n(101, 110)$, for each $i = 2, 3, \dots, n-2$ in turn with $u := w_{i+1}$, if $w_i > u$ and $w_j = u$ for some $j > i+1$, then replace w_{i+1} with w_i .

2.3.4. h_6 . A nonnegative sequence $w = (w_i)_{i=1}^n$ avoids 101,102, and 201 if and only if it is either weakly increasing or weakly decreasing after the first descent. Thus, if the first descent top is $w_k = s$, we have $1 \leq s < k \leq n-1$, and $(w_i)_{i=k+1}^n$ is weakly decreasing with $w_{k+1} \leq s-1$. There are $\binom{n-k+s-1}{s-1}$ such sequences $(w_i)_{i=k+1}^n$. There are $\binom{k-1}{s}$ weakly increasing ascent sequences of length k that end with an s , and 2^{n-1} weakly increasing ascent sequences of length n . Hence,

$$|A_n(101, 102, 201)| = 2^{n-1} + \sum_{k=2}^{n-1} \sum_{s=1}^{k-1} \binom{k-1}{s} \binom{n-k+s-1}{s-1} = F_{2n-1}.$$

2.3.5. h_7 and h_8 . Ascent sequences avoiding (101,110) were counted in [1] using generating trees. Here we give a direct count. Consider the position $k \geq 1$ of the last 0 in a (101,110)-avoiding ascent sequence w .

If $k = 1$ then w_2 (if present) is 1 and $(w_i - 1)_{i \geq 2}$ is easily seen to be an arbitrary (101,110)-avoider.

If $k \geq 2$, write w as uv where u consists of the first k entries of w . Then $u_k = 0$ and the nonzero entries of u are all distinct since u avoids 110 and must form, left to right, an initial segment of the positive integers since u is an ascent sequence. Thus each interior entry of u is 0 or 1 more than the largest previous entry, giving 2^{k-2} choices for u . Let $m = \max u$. Then v_1 (if present) must be $m + 1$. This is because $v_1 \leq m + 1$ since u contains only m ascents and so $v_1 = m + 1$ to avoid 101. Also, $v - (m + 1)$ (termwise subtraction) is an arbitrary (101,110)-avoider.

In any case $w \in A_n(101, 110)$ decomposes as an ordered pair (u, v') where u, v' are (101,110)-avoiders whose lengths sum to n , u is nonempty of length $k \geq 1$ with $\max(1, 2^{k-2})$ choices for u , and v' may be empty. Iterating the decomposition, we find, with $\mathcal{C}_n$ denoting the set of compositions $c = (c_i)_{i=1}^r$ into positive parts c_i ,

$$|A_n(101, 110)| = \sum_{c \in \mathcal{C}_n} \prod_{i=1}^r \max(1, 2^{c_i-2}) = F_{2n-1}.$$

The preceding observations show that (101,110)-avoiders also avoid 201 and 210. Thus h_7 -avoiders and h_8 -avoiders both coincide with (101,110)-avoiders.

2.4. Proof of Theorem 1 part (4).

2.4.1. i_1 . Let $w \in A_n(000, 021, 101)$. Since an ascent sequence begins with a 0, to avoid 021 implies that only 0 can be a descent bottom, and so avoidance of 000 implies w has at most one descent. Let $k \geq 1$ denote the position of the last 0 in w . If $k = 1$, then w is weakly increasing and so has the form $0v$ where $v - 1 \in \mathcal{F}_{n-1}$ (where $\mathcal{F}$ is defined in Section 2.1.2 above) since w is an ascent sequence and to avoid 000. If $k = 2$, then $w = 00v$ where $v - 1 \in \mathcal{F}_{n-2}$. These two cases contribute $F_n + F_{n-1}$ to the count. Now suppose $k \geq 3$ and write w as $0u0v$. Here $u - 1 \in \mathcal{F}_{k-2}$ and $v - (m + 1) \in \mathcal{F}_{n-k}$ where $m := \max(u)$. Hence

$$(2.2) \quad |A_n(000, 021, 101)| = F_n + \sum_{k=2}^n F_{k-1} F_{n-k+1}.$$

2.4.2. i_2 . Let $w \in A_n(000, 021, 110)$. Again, let $k \geq 1$ denote the position of the last 0 in w . Again, if $k = 1$, then w is weakly increasing and so has the form $0v$ where $v - 1 \in \mathcal{F}_{n-1}$, and if $k = 2$, then $w = 00v$ where $v - 1 \in \mathcal{F}_{n-2}$. If $k = n \geq 3$, then $w = 01 \dots (n-2)0$ in order to avoid 110. Now suppose $3 \leq k \leq n-1$, and write $w = 0u0v$. Then $u = 12 \dots (k-2)$ and $v = (v_i)_{i=1}^{n-k}$ is weakly increasing and satisfies $v_{i+1} \leq v_i + 1$ for all i if $v_1 = k-1$. But v_1 may be $k-2$ in which case the inequality $v_{i+1} \leq v_i + 1$ may be violated with a jump of *two* units, $v_{i+1} = v_i + 2$, for at most one value of i due to the extra ascent introduced by v_1 .

First, suppose no such jump occurs. If $v_1 = k-2$, then v_2 (if present) must be $k-1$ and $(v_i - (k-1))_{i=2}^{n-k} \in \mathcal{F}_{n-k-1}$, counted by F_{n-k} . Otherwise, $v_1 = k-1$ and $(v_i - (k-1))_{i=1}^{n-k} \in \mathcal{F}_{n-k}$, counted by F_{n-k+1} . Now suppose such a jump occurs starting at position r , $1 \leq r \leq n-k-1$.

Then $k \leq n - 2$, $v_1 = k - 2$, $v_2 = k - 1$ and $(v_i - (k - 1))_{i=2}^r \in \mathcal{F}_{r-1}$, counted by F_r . Also, $(v_i - (v_r + 2))_{i=r+1}^{n-k} \in \mathcal{F}_{n-k-r}$, counted by $F_{n-k-r+1}$. Hence, for $n \geq 3$,

$$|A_n(000, 021, 110)| = F_n + F_{n-1} + 1 + \sum_{k=3}^{n-1} (F_{n-k} + F_{n-k+1}) + \sum_{k=3}^{n-2} \sum_{r=1}^{n-1-k} F_r F_{n-k-r+1},$$

which can be shown to agree with the expression in (2.2).

3. FURTHER RESULTS: QUADRUPLES OF 3-LETTER PATTERNS

As a consequence of the results of the previous section, here we present several pattern avoidance classes of ascent sequences avoiding a quadruple of length-3 patterns whose counting sequences are closely related to Fibonacci numbers.

Proposition 4. *For each of the following sets of 4 patterns f , we have $|A_n(f)| = F_{n+1}$.*

$$\begin{array}{llll} \{000, 001, 100, 101\} & \{000, 001, 100, 102\} & \{000, 001, 100, 201\} & \{000, 001, 101, 102\} \\ \{000, 001, 101, 201\} & \{000, 001, 102, 201\} & \{000, 010, 021, 100\} & \{000, 010, 021, 101\} \\ \{000, 010, 021, 102\} & \{000, 010, 021, 110\} & \{000, 010, 021, 120\} & \{000, 010, 021, 201\} \\ \{000, 010, 021, 210\} & \{000, 010, 100, 101\} & \{000, 010, 100, 102\} & \{000, 010, 100, 110\} \\ \{000, 010, 100, 120\} & \{000, 010, 100, 201\} & \{000, 010, 100, 210\} & \{000, 010, 101, 102\} \\ \{000, 010, 101, 110\} & \{000, 010, 101, 120\} & \{000, 010, 101, 201\} & \{000, 010, 101, 210\} \\ \{000, 010, 102, 110\} & \{000, 010, 102, 120\} & \{000, 010, 102, 201\} & \{000, 010, 102, 210\} \\ \{000, 010, 110, 120\} & \{000, 010, 110, 201\} & \{000, 010, 110, 210\} & \{000, 010, 120, 201\} \\ \{000, 010, 120, 210\} & \{000, 010, 201, 210\} & & \end{array}$$

Proof. Each of these quadruples contains either the pair $\{000, 001\}$ or the pair $\{000, 010\}$. So the results follow from Section 2.1. $\square$

Proposition 5. *For all $n \geq 5$, $|A_n(000, 021, 102, 120)| = F_{n+1}$.*

Proof. For $n \geq 5$, there can be no descent to 0 (to avoid 120). Thus for $n \geq 5$, these avoiders coincide with $\mathcal{F}_n$, defined in Section 2.1.2, counted by F_{n+1} . $\square$

Proposition 6. *For all $n \geq 5$, $|A_n(000, 021, 102, 110)| = F_{n+1} + 1$.*

Proof. For $n \geq 5$, these avoiders include case Proposition 5 and also $012 \dots (n-1)0$, counted by $F_{n+1} + 1$. $\square$

Proposition 7. *For all $n \geq 5$,*

$$\begin{aligned} |A_n(000, 021, 100, 102)| &= |A_n(000, 021, 102, 201)| = |A_n(000, 021, 102, 210)| \\ &= L_n = F_{n+1} + F_{n-1}, \end{aligned}$$

where L_n is the n th Lucas number, sequence A000032 in [10].

Proof. Let $\tau = \{000, 021, 102\}$. For a τ -avoider, all descent bottoms must be 0, to avoid 021, and so an avoider has at most one descent, to avoid 000. Furthermore, this descent bottom must end the sequence, to avoid 021 and 102. Hence, with m' denoting $\max(w')$ and $w \in A_n(\tau)$, either (i) $w \in \mathcal{F}_n$, or (ii) $w = 0w'(m' + 1)0$ with $w' - 1 \in \mathcal{F}_{n-3}$, or (iii) $w = 0w'(m' + 1)(m' + 1)0$ with $w' - 1 \in \mathcal{F}_{n-4}$ (here the word $w' - 1$ is obtained by the word w' by decreasing each letter by 1). Thus τ -avoiders are counted by $F_{n+1} + F_{n-2} + F_{n-3} = L_n$, the Lucas number, for $n \geq 5$. It is clear that $(000, 021, 102)$ -avoiders also avoid 100, 201, and 210, and so all 3 quadruples here give rise to Lucas numbers. $\square$

Proposition 8. *For all $n \geq 1$,*

$$\begin{aligned} |A_n(000, 021, 101, 110)| &= |A_n(000, 101, 102, 110)| = |A_n(001, 100, 101, 102)| \\ &= |A_n(001, 100, 101, 201)| = |A_n(001, 100, 102, 201)| = F_{n+2} - 1. \end{aligned}$$

Proof. An avoider $w \in A_n(000, 021, 101, 110)$ is either in $\mathcal{F}_n$ or begins $012 \dots (m-2)0$ with a descent to 0 at position $m \in [3, n]$, and $(w_i - (m-1))_{i=m+1}^n \in \mathcal{F}_{n-m}$. So, for $n \geq 1$,

$$|A_n(000, 021, 101, 110)| = |\mathcal{F}_n| + \sum_{m=3}^n |\mathcal{F}_{n-m}| = F_{n+1} + \sum_{m=3}^n F_{n-m+1} = F_{n+2} - 1.$$

For an avoider $w \in A_n(000, 101, 102, 110)$ either $w \in \mathcal{F}_n$ or the last entry of w is a descent bottom that is larger than the last repetition (successive equal entries) in w , to avoid 110. In this case either $w = 012 \dots (n-2)a$ with no repetitions and $a \in [0, n-3]$, giving $n-2$ choices when $n \geq 2$, or (iii) the last repetition ends at position $k \in [2, n-3]$ and w has the form $w = w'mm(m+1)(m+2) \dots (m+n-k-2)a$ with the second m in position k , $w' \in \mathcal{F}_{k-2}$ and $m-1$ the last entry of w' , and $a \in [m+1, m+n-k-2]$. This gives $\binom{m}{k-2-m}$ choices for w' and $n-k-2$ choices for a . Hence, for $n \geq 2$,

$$\begin{aligned} |A_n(000, 101, 102, 110)| &= F_{n+1} + n - 2 + \sum_{k=2}^{n-3} (n-k-2) \sum_{m=\lceil k/2 \rceil - 1}^{k-2} \binom{m}{k-2-m} \\ &= F_{n+2} - 1. \end{aligned}$$

The last 3 quadruples are covered by Section 2.2. $\square$

Proposition 9. *For all $n \geq 1$, $|A_n(000, 101, 110, 120)| = F_{n+2}^{(3)}$, the $(n+2)$ -Tribonacci number, sequence A000073 in [10].*

Proof. Let $u_n = |A_n(\tau)|$ where $\tau = \{000, 101, 110, 120\}$ and for an avoider w , let $L_0(w)$ denote the position of the last 0 in w . If $L_0(w) = 1$, then $(w_2 - 1) \dots (w_n - 1) \in A_{n-1}(\tau)$. If $L_0(w) = 2$, then $w_1 = w_2 = 0$ and $(w_3 - 1) \dots (w_n - 1) \in A_{n-2}(\tau)$. Otherwise, since all descents are of precisely one unit to avoid 120, $L_0(w) = 3$ and $w_1 w_2 w_3 = 010$ and $(w_4 - 2) \dots (w_n - 2) \in A_{n-3}(\tau)$. Hence, the sequence u_n satisfies the “tribonacci” recurrence $u_n = u_{n-1} + u_{n-2} + u_{n-3}$.

Here is an explicit counting approach. If $w \in A_n(000, 101, 110, 120)$ has no weak descents, then $w = 012 \dots (n-1)$. Otherwise, a typical avoider looks like

$$012343|56788|9\ 10\ 9|11\ 11|12\ 13\ 14\ 15,$$

where a divider has been inserted after each weak descent bottom to divide the ascent sequence into convenient blocks. The block after the last weak descent bottom forms an interval of integers (possibly empty). Every block other than this one has successive entries increasing by 1 except the last entry may be an equality or (but only in blocks of length ≥ 3) a decrease of 1. Furthermore, the first entry of each noninitial block is 1 more than the maximum entry of the previous block. The restricted increases are due to the ascent sequence requirement and then the other restrictions are to avoid 000, 101, 110, and 120.

Consequently, if w has r ($0 \leq r \leq n-2$) entries after the last weak descent, then w is determined by a composition of $m := n-r$ in which all parts are ≥ 2 and each part ≥ 3 is colored red (R) or blue (B), according as the last entry of the corresponding block is an equality or a decrease. For the example above, the composition of $m = 16$ is $(6_B, 5_R, 3_B, 2)$. The number of such colored compositions of m is sequence A001590 in [10], $(0, 1, 2, 3, 6, 11, 20, 37, \dots)_{m \geq 1}$. The desired counting sequence can now be obtained from the partial sums of this sequence. $\square$

Proposition 10. *For all $n \geq 1$,*

$$\begin{aligned} |A_n(000, 021, 100, 101)| &= |A_n(000, 021, 100, 110)| = |A_n(000, 021, 101, 201)| \\ &= |A_n(000, 021, 101, 210)| = |A_n(000, 021, 110, 201)| = |A_n(000, 021, 110, 210)| = \\ &= |A_n(000, 101, 102, 210)| = \frac{(n+4)F_n + 2nF_{n-1}}{5}. \end{aligned}$$

Proof. Ascent sequences avoiding 000, 021, and 101 are characterized in Section 2.4.1. From the characterization, it is clear that such sequences also avoid 100, 201, 210 and so the first, third, and fourth quadruples all have avoiders coinciding with $A_n(000, 021, 101)$, counted in Section 2.4.1. Similarly, ascent sequences avoiding 000, 021, and 110 are characterized in Section 2.4.2 and it is clear that such sequences also avoid 100, 201, and 210, and so the second, fifth, and sixth quadruples all have avoiders coinciding with $A_n(000, 021, 110)$, counted in Section 2.4.2.

Now let $\tau = \{000, 101, 102, 210\}$, the seventh quadruple. It is convenient to set $\mathcal{F}_{n,m} = \{w \in \mathcal{F}_n : w_n = m\}$. Note that $|\mathcal{F}_{n,m}| = \binom{m+1}{n-m-1}$ and $w \in \mathcal{F}_{n,m}$ contains $n-1-m$ repeated letters. After a descent $v > u$ in a τ -avoider, all later entries must lie in $[u, v-1]$ and any such entry would repeat an earlier entry and thereby introduce a 000 or 101. Hence, only the last entry can be a descent bottom. So for $w \in A_n(\tau)$, either $w \in \mathcal{F}_n$ or w has the form (i) $w = w'ma$ where $w' = 0 \dots m-1 \in \mathcal{F}_{n-2,m-1}$ and $a \in [0, m-1]$ is not a repeated letter in w' , to avoid 000, or (ii) $w = w'mma$ where $w' = 0 \dots m-1 \in \mathcal{F}_{n-3,m-1}$ and again, $a \in [0, m-1]$ is not a repeated letter in w' . In case (i), there are $|\mathcal{F}_{n-2,m-1}| = \binom{m}{n-2-m}$ choices for w' ,

denotes the set of compositions $(c_i)_{i=1}^k$ of m into k nonnegative parts,

$$\begin{aligned} |A_n(000, 101, 120, 210)| &= \sum_{\substack{a, b \geq 0 \\ 3a+4b \leq n}} \binom{a+b}{b} \prod_{c \in \mathcal{C}_{n-3a-4b, a+b+1}} F_{c_i+1} \\ &= \sum_{\substack{a, b \geq 0 \\ 3a+4b \leq n}} \binom{a+b}{b} [x^{n-3a-4b}] \left(\frac{1}{1-x-x^2} \right)^{a+b+1}, \end{aligned}$$

where $[x^m]$ extracts the coefficient of x^m , since $\frac{1}{1-x-x^2}$ is the generating function $\sum_{n \geq 0} F_{n+1} x^n$. □

Proposition 12. *For all $n \geq 1$,*

$$\begin{aligned} |A_n(021, 101, 201, 210)| &= |A_n(021, 120, 201, 210)| = |A_n(100, 101, 201, 210)| \\ &= |A_n(101, 110, 201, 210)| = F_{2n-1}. \end{aligned}$$

Proof. The avoiders for these four quadruples coincide, respectively, with the avoiders for cases h_1 , h_3 , h_5 , and h_7 in Section 2.3. This is clear from the descriptions of the avoiders in Section 2.3. □

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