Pattern avoidance in circular permutations

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In the present context, a *circular permutation* is an arrangement of distinct positive integers (called letters) clockwise around a circle, two arrangements being considered the same if they differ only by a rotation. A (linear) permutation is a list of distinct positive integers and its *reduced form* is obtained by replacing its smallest entry by 1, its next smallest by 2, and so on. A *pattern* is any such reduced form. An occurrence of a pattern τ in a cyclic permutation π is a sequence of letters in clockwise order in π (not extending more than one revolution) whose reduced form is τ. We will always represent a circular permutations on $[n] := \{1, 2, \ldots, n\}$ as a linear permutation that ends in n. For example, 5 6 4 2 3 1 7 has just one occurrence of the pattern 1234, the letters being 2 3 5 6 (keep in mind we may wrap around from the end to the start).

Much work has been done counting permutations with restrictions on the patterns they contain and possibly other restrictions as well; see [1] for a survey. There are two 3-letter patterns up to cyclic rotation: 123 and 132. Only the identity circular permutation avoids 132 and only the reverse identity avoids 123. Here we find simple explicit formulas for the number of circular permutations avoiding a single four letter pattern. There are 6 such patterns up to cyclic rotation and "reversing" cuts the number to 3: 1234, 1324 and 1342 are representative. In each case, an analysis of the structure according to the position of $n-1$ (recall n is at the end) yields a recurrence relation. For completeness we include a proof of the following result.

Proposition 1 ([2]). *There are* 2^{n-1} *(linear) permutations on* $[n]$ *that avoid both* 213 *and* 231.

Proof. Consider such a permutation π and any letter a in π. The letters following a in π must either all be $> a$ or all be $< a$, else a would be the first letter of a proscribed pattern. So the the first letter of π is 1 or n and we have two choices for each successive letter except the last. This observation translates into a bijection from 0-1 sequences of length $n-1$ to permutations on $[n]$ that avoid 213 and 231: given $(b_i)_{i=1}^{n-1}$, $b_i = 0$ or 1, form π left to right as follows. For $i = 1$ to $n-1$ in turn, if $b_i = 1$ (resp. 0), set $\pi_i =$ the largest (resp. smallest) letter in $[n]$ that has not already been used in π. For example, with $n = 8$, $(0,1,1,1,0,1,0) \to (1,8,7,6,2,5,3,4)$. □

Theorem 1. *The number of* 1324*-avoiding circular permutations on* $[n]$ *is the Fibonacci number* F_{2n-3}

Proof. Let u_n denote the desired number. Then $u_1 = u_2 = 1$, $u_3 = 2$. For $n \geq 4$, we count by the location $k \in [n-1]$ of $n-1$ in π. First suppose $k \leq n-2$ so that $n-1$ and n are not adjacent. The first $k-1$ letters in π must all be $>$ than all the letters between $n-1$ and n, else there would be a 1324 pattern of the form $a\ n{-}1\ b\ n$. So the first k entries are $\{n-j\}_{j=1}^{k}$. Furthermore, they must occur in increasing order, else there would be a $c\ b\ n{-}1\ a$ pattern, rotationally equivalent to a 1324. We leave the reader to verify that any such circular permutation is in fact 1324-avoiding. It is now easy to check that for $k \leq n-2$ (including $k = 1$), deleting the first k entries and reducing (in this case, reducing means replacing n by $n-k$) is a bijection to 1324-avoiding circular permutations on $[n-k]$, counted by u_{n-k}. Now suppose $k = n-1$. Deleting the n gives a bijection to the $n-1$ case, counted by u_{n-1}. Summing over k, we get a recurrence relation: $u_n = u_{n-1} + \sum_{k=1}^{n-2} u_{n-k}$ $n \geq 3$ with initial condition $u_1 = u_2 = 1$, whose solution is $u_n = F_{2n-3}$.

Here is a short combinatorial proof that $u_n := F_{2n-3}$ satisfies the recurrence. Let $\mathcal{F}_n$ denote the set of 0-1 sequences of length $n-1$ that start with 0 (unless they're empty) and do not contain two consecutive 1's, so that $|\mathcal{F}_n| = F_n$. Set $\mathcal{G}_n = \mathcal{F}_{2n-3}$ so that $|\mathcal{G}_n| = u_n$. If $w \in \mathcal{G}_n$ begins $01\ldots$, deleting this 01 is a bijection to $\mathcal{G}_{n-1}$. Otherwise w begins $00\ldots$ and we count by number k of consecutive 10 pairs immediately following the initial 00. Clearly, $k \in [0, n-3]$ and deleting the initial 00 and all k of these 10's is a bijection to $\mathcal{G}_{n-1-k}$, counted by u_{n-1-k}. □

Theorem 2. *There are* $2^{n-1} - (n-1)$ 1342*-avoiding circular permutations on* $[n]$.

Proof. Let u_n denote the number of 1342-avoiding circular permutations on $[n]$. Once again $u_1 = u_2 = 1,\ u_3 = 2$, so let $n \geq 4$. If $n-1$ is in first position, deleting n (and rotating) gives a bijection to 1342-avoiding circular permutations on $[n-1]$, counted by u_{n-1}.

Now suppose $n-1$ is in position k with $k \in [2, n-2]$. Write $\pi = w_1\ n-1\ \ldots\ n$ with w_1 the initial subword of length $k-1$. Then w_1 is nonempty and its letters are increasing, else $n-1, n$ would be the middle letters of a 1342 pattern. Also, the letters in w_1 form an interval of consecutive integers. To see this, suppose $a < c$ are letters of w_1 and $b \in [a+1, c-1]$. Then b can't precede a in π or fall between c and $n-1$ because the letters in π increase. Neither can c fall between $n-1$ and n, else $a\ c\ n-1\ b$ would be a 1342. So b must fall between a and c and w_1 is a list of consecutive integers.

It follows that w_1 splits the remaining letters in $[n-2]$ into two sets A, B (either or both may be empty, A denotes the set of smaller letters). Write π as $w_1\ n-1\ w_0\ n$ so that w_0 is comprised of the letters in $A \cup B$. The letters in A are in increasing order in π because if $w_0 = \ldots\ a_2\ \ldots\ a_1\ \ldots$ with $a_1 < a_2$, then $a_1\, b\ n-1\ a_2$ would be a 1342 for any $b \in w_1$. All letters of C precede all letters of A because if $w_0 = \ldots\ a\ \ldots\ c\ \ldots$ with $a \in A$, $c \in C$, then $a\, c\, n\, b$ would be a 1342 for any $b \in w_1$.

So far, then, we have shown that π has the form

$$w_1\ n-1\ w_2\ w_3\ n$$

with w_1 of length $k-1$ (and so nonempty), all letters in $w_2 >$ those in w_1, all letters in w_3 increasing and $<$ those in w_1. Now we claim further that the word w_2 must avoid the patterns 213 and 231. Indeed, if $w_2 = \ldots\, b\, \ldots\, a\, \ldots\, c\, \ldots$ with $a < b < c$, then $a\, c\, n\, b$ is a 1342 in π. And if $w_2 = \ldots\, b\, \ldots\, c\, \ldots\, a\, \ldots$, then $x\, b\, c\, a$ is a 1342 for any x in w_1. We leave the reader to verify that any circular permutation with $n-1$ in position $k \in [2, n-2]$ that meets all these specifications is in fact 1342-avoiding. Let us count these latter permutations. The initial segment w_1 is determined by its first letter j which must lie in $[n-k]$. Then w_3 is fully determined as are the letters of w_2. Since w_2 avoids 213 and 231 (Prop. 1), there are $2^{|w_2|-1} = 2^{n-k-j-1}$ possibilities for w_2 unless $j = n-k$ in which case w_2 is empty and there is just one possibility. Summing over j, we have $1 + \sum_{j=1}^{n-k-1} 2^{n-k-j-1} = 2^{n-k-1}$ possibilities in all. Now summing over k, we get the

defining recurrence $u_1 = 1,\ u_2 = 1,\ u_n = u_{n-1} + 2^{n-2} - 1$ for $n \geq 3$, with solution $u_n = 2^{n-1} - (n-1)$ valid for all $n \geq 1$.

The preceding analysis could be worked up into a bijection. Briefly, let $\mathcal{A}_n$ denote the set of n-bit 0-1 sequences that do not contain precisely one 1 so that $|\mathcal{A}_n| = 2^n - n$. Given $u \in \mathcal{A}_{n-1}$ build up $\pi = w_1\ n{-}1\ w_2\ w_3\ n$ as follows. If $u_1 = 0$, put $n-1$ in position 1 and proceed recursively. Otherwise, $u_1 = 1$ and put $n-1$ in the position $k \geq 2$ of the next 1 in u (there will be one!). Then let $j \in [n-k]$ be the position of the next 1 after that in $u_{k+1} \ldots u_{n-1}$ (with $j = n-k$ if there are no more 1's). Use j as the initial letter that determines w_1 and use $u_{k+j+1} \ldots u_{n-1}$ to construct w_2 (Prop. 1). □

Theorem 3. *There are* $2^n + 1 - 2n - \binom{n}{3}$ 1234*-avoiding circular permutations on* $[n]$.

Proof. Again let u_n denote the desired number and k the position of $n-1$. If $k = 1$, deleting the $n-1$ and reducing is a bijection to the $n-1$ case. If $k = 2$, write π as $a\ n{-}1\ w_1\ n$ with $a \in [n-2]$. All letters $> a$ in w_1 clearly occur in decreasing order, as must all letters $< a$ (else $n-1$ would terminate a 1234). So the only freedom is in placement of the letters $< a$, giving $\sum_{a=1}^{n-2} \binom{n-3}{a-1} = 2^{n-3}$ possibilities with $k = 2$.

Now suppose $k \geq 3$ so that $\pi = w_1\ n{-}1\ w_2\ n$ with $|w_1| = k-1 \geq 2$. Let m and M denote the minimum and maximum letters in w_1 and let A consist of the remaining $k-3$ letters in w_1. Then the letters of w_2 can be partitioned into three sets: $B = [1, m-1]$, $C = [m+1, M-1] \backslash A$, $D = [M+1, n-2]$ of sizes $|B| = m-1$, $|C| = M-m-1-K$, $|D| = n-M-2$ respectively, where $K = |A| = k-3$. All letters $< M$ in w_2 must occur in decreasing order, else there would be a 1234 terminating in $M\ n{-}1$; likewise for all letters $> m$, else $muvn$ would be a 1234 for some $u, v \in w_2$. This means that unless $M = m+1$, *all* letters in w_2 must decrease and in case $M = m+1$ (and hence C is empty) the letters of B and D separately must occur in decreasing order in w_2 giving $\binom{n+m-M-3}{m-1}$ choices to place the letters of B and D in w_2. Again we leave the reader to verify that if all these conditions are met, the proscribed pattern is avoided.

Using standard binomial coefficient identities, we can now count possibilities. Fix $k \in [3, n-1]$ (so $K = k-3$ is also fixed) and sum over m and M. There are $\binom{M-m-1}{K}$ ways to select the set A. For $C = \emptyset$ (in which case $M = m+1+K$), the count is

$$\sum_{m=1}^{n-K-3} \binom{n-K-4}{m-1} = 2^{n-K-4} = 2^{n-1-k}.$$

For $C \neq \emptyset$, the count is

$$\begin{aligned}\sum_{m=1}^{n-K-4} \sum_{M=m+K+2}^{n-2} \binom{M-m-1}{K} &= \sum_{m=1}^{n-K-4} \left(\binom{n-m-2}{K+1} - 1 \right) \\ &= \binom{n-2}{K+2} - n + K + 3 \\ &= \binom{n-2}{k-1} - n + k\end{aligned}$$

Taking the cases $C = \emptyset$ and $C \neq \emptyset$ together, the total count for fixed $k \in [3, n-1]$ is $2^{n-1-k} + \binom{n-2}{k-1} - n + k$, while, as shown above, for $k = 2$ it is 2^{n-3} and for $k = 1$ it is u_{n-1}. Summing over k now yields the recurrence relation $u_1 = u_2 = 1$, $u_n = u_{n-1} + 2^{n-1} - n - \binom{n-2}{2}$ for $n \geq 3$, with solution $u_n = 2^n + 1 - 2n - \binom{n}{3}$. □

We conclude by noting that the preceding exact enumerations yield growth constants ($\lim_{n\to\infty} u_n^{1/n}$) for the patterns 1324, 1342 and 1234 of $\phi^2 \approx 2.618$, 2 and 2 respectively, where ϕ denotes the golden ratio.

References

[1] Herbert S. Wilf, The patterns of permutations, *Discrete Math.* **257** (2002) 575–583.

[2] R. Simion and F. W. Schmidt, Restricted permutations, *Europ. J. Combinatorics* **6** (1985) 383-406.