On Conjugates for Set Partitions and Integer Compositions

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Abstract

There is a familiar conjugate for integer partitions: transpose the Ferrers diagram, and a conjugate for integer compositions: transpose a Ferrers-like diagram. Here we propose a conjugate for set partitions and exhibit analogous pairs of statistics interchanged by the conjugate on set partitions and integer compositions respectively.

0 The Conjugate of an Integer Partition

A partition of n is a weakly decreasing list of positive integers, called its parts, whose sum is n. The Ferrers diagram of a partition $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is the k-row left-justified array of dots with a_i dots in the i-th row. The conjugate, obtained by transposing the Ferrers diagram, is a well known involution on partitions of n that interchanges the largest part and the number of parts.

1 A Conjugate for Set Partitions

The partitions of an n-element set, say $[n] = \{1, 2, \ldots, n\}$, into nonempty blocks are counted by the Bell numbers, A000110 in OEIS. A *singleton* is a block containing just 1 element and an *adjacency* is an occurrence of two consecutive elements of $[n]$ in the same block. Consecutive is used here in the cyclic sense so that n and 1 are also considered to be consecutive (the ordinary sense is considered in [2]). We say i *initiates* an adjacency if i and $i+1 \bmod n$ are in the same block and analogously for terminating an adjacency. The number of k-block partitions of $[n]$ containing *no* adjacencies is considered in a recent Monthly problem proposed

by Donald Knuth [3]. Suppressing the number of blocks, it turns out that # singletons and # adjacencies have the same distribution.

Theorem 1. *There is a bijection ϕ on partitions of $[n]$ that interchanges number of singletons and number of adjacencies.*

To prove this, we need to consider partitions on arbitrary subsets rather than just initial segments of the positive integers. (I thank Robin Chapman [1] for pointing out that my original proof was incorrect.) The notions of adjacency, adjacency initiator and adjacency terminator generalize in the obvious way. We always write a partition with elements increasing within each block and blocks arranged in increasing order of their first (smallest) elements. Thus, for example, with a dash separating blocks, the partition $\pi = 3\ 5\ 12$ - $4\ 8\ 10$ - 7 has support $\mathrm{supp}(\pi) = \{3, 4, 5, 7, 8, 10, 12\}$ and adjacencies $(8, 10)$, $(12, 3)$. In a partition with one-element support, this element is considered to be both an adjacency initiator and an adjacency terminator.

Consider the operation SeparateIS on positive-integer-support partitions defined by $\mathrm{SeparateIS}(\pi) = \big(\rho, (I, S)\big)$ where I is the set of adjacency initiators of π, S is the set of singleton elements of π, and ρ is the partition obtained by suppressing the elements of $I \cup S$ in π. Thus, for $\pi = 3\ 5\ 12$ - $4\ 8\ 10$ - 7, we have $I = \{8, 12\}$, $S = \{7\}$ and $\rho = 3\ 5$ - $4\ 10$. Also, for a one-block partition $\pi = a_1 a_2 \ldots a_k$, $\mathrm{SeparateIS}(\pi) = \big(\epsilon, (\{a_1, \ldots, a_k\}, \emptyset)\big)$ where ϵ denotes the empty partition, unless $k = 1$ in which case it is $\big(\epsilon, (\{a_1\}, \{a_1\})\big)$. Clearly, for ρ a partition and A, B finite sets of positive integers, the pair $\big(\rho, (A, B)\big)$ lies in the range of SeparateIS iff (i) $\rho = \epsilon$, $A = B$ and both are singletons, or (ii) $\mathrm{supp}(\rho)$, A, B are disjoint and for no successive pair (a, b) in $\mathrm{supp}(\rho) \cup A \cup B$ is $a \in A$ and $b \in B$ (the successor of an adjacency initiator cannot be a singleton).

Analogously, define SeparateST with S the set of singleton elements and T the set of adjacency terminators. Note that the condition for $\big(\rho, (A, B)\big)$ to lie in range(SeparateST) is precisely the same as for it to lie in range(SeparateIS). Both are injective and so we may define their respective inverses CombineIS and CombineST and we will make use of the crucial property that these inverses have identical domains. For example, CombineST is defined on $\big(3\ 10$ - $4\ 7$ - $12\ , (\{11\}, \{1, 2\})\big)$ and yields the partition $1\ 2\ 12$ - $3\ 10$ - $4\ 7$ - 11.

Now we can define the desired bijection. Given a partition π on $[n]$, form a sequence $\big(\rho_1, (I_1, S_1)\big)$, $\big(\rho_2, (I_2, S_2)\big)$, $\ldots$, $\big(\rho_k, (I_k, S_k)\big)$ by setting $\big(\rho_1, (I_1, S_1)\big) = \mathrm{SeparateIS}(\pi)$, $\big(\rho_2, (I_2, S_2)\big) = \mathrm{SeparateIS}(\rho_1)$, $\ldots$, $\big(\rho_k, (I_k, S_k)\big) = \mathrm{SeparateIS}(\rho_{k-1})$ stopping when ρ_k has no adjacency initiators and no singletons (as must eventually occur). For example,

with $n = 12$ and $\pi = 1$ - 2 - 3 11 12 - 4 7 10 - 5 9 6 8, the results are laid out in the following table

j	ρ_j	I_j	S_j
1	3 12 – 4 7 10 – 5 9 – 6 8	$\{11\}$	$\{1,2\}$
2	3 – 4 7 10 – 5 9 – 6 8	$\{12\}$	$\emptyset$
3	4 7 10 – 5 9 – 6 8	$\emptyset$	$\{3\}$
4	4 7 – 5 9 – 6 8	$\{10\}$	$\emptyset$

and ρ_4 has no adjacency initiators (hence, no adjacencies) and no singletons. Next, form a sequence of partitions $\tau_k, \tau_{k-1}, \ldots, \tau_1, \tau_0$ by reversing the procedure but using CombineST rather than CombineIS. More precisely, set $\tau_k = \rho_k$, τ_{k-1} = CombineST on $\big(\tau_k, (I_k, S_k)\big)$, τ_{k-2} = CombineST on $\big(\tau_{k-1}, (I_{k-1}, S_{k-1})\big)$, $\ldots, \tau_0$ = CombineST on $\big(\tau_1, (I_1, S_1)\big)$. Note that for $j = k, k-1, \ldots, 1$ in turn, CombineST is defined on $\big(\tau_j, (I_j, S_j)\big)$ because (i) $\mathrm{supp}(\tau_j) = \mathrm{supp}(\rho_j)$, (ii) CombineIS is certainly defined on $\big(\rho_j, (I_j, S_j)\big)$, and (iii) CombineST, CombineIS have the same domain. The example yields

j	τ_j	I_j	S_j
4	4 7 – 5 9 – 6 8	$\{10\}$	$\emptyset$
3	4 7 – 5 9 – 6 8 – 10	$\emptyset$	$\{3\}$
2	3 10 – 4 7 – 5 9 – 6 8	$\{12\}$	$\emptyset$
1	3 10 – 4 7 – 5 9 – 6 8 – 12	$\{11\}$	$\{1,2\}$
0	1 2 12 – 3 10 – 4 7 – 5 9 – 6 8 – 11		

Now $\phi : \pi \to \tau_0$ is the desired bijection: ϕ is clearly reversible and sends adjacency initiators to singletons and singletons to adjacency terminators, and hence interchanges # adjacencies and # singletons. □

The *complement* of a partition π on $[n]$ is $n + 1 - \pi$ (elementwise). Our proposed conjugate is as follows.

Definition *The* conjugate *of a partition π on $[n]$ is obtained by applying the map ϕ followed by complementation.*

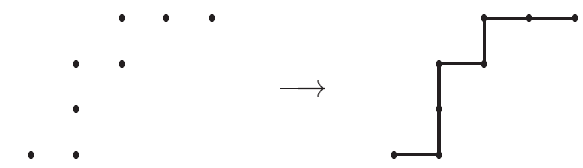
Since complementation is an involution and sends adjacency initiators to adjacency terminators and vice versa, it is not hard to see that conjugation is an involution on partitions of $[n]$ that interchanges # singletons and # adjacencies.

2 The Conjugate of an Integer Composition

A *composition* of n is a list of positive integers—its parts—whose sum is n. There is a bijection from compositions of n to subsets of $[n-1]$ via partial sums: $(c_i)_{i=1}^{k} \mapsto \{\sum_{j=1}^{k} c_j\}_{i=1}^{k-1}$, and a further bijection from subsets S of $[n-1]$ to lattice paths of $n-1$ unit steps North (N) or East (E): the ith step is N if $i \in S$ and E otherwise. The conjugate of a composition is defined by: pass to lattice path, flip the path in the 45° line, and pass back. For example, with $n = 8$ and $k = 4$,

$$\underset{\text{composition}}{(2,1,2,3)} \to \underset{\text{subset}}{\{2,3,5\}} \to \overset{\text{path}}{\underset{1\ 2\ 3\ 4\ 5\ 6\ 7}{ENNENEE}} \overset{\text{flip}}{\to} \overset{\text{path}}{\underset{1\ 2\ 3\ 4\ 5\ 6\ 7}{NEENENN}} \to \underset{\text{subset}}{\{1,4,6,7\}} \to \underset{\text{conjugate composition}}{(1,3,2,1,1)}.$$

There is a neat graphical construction for the lattice path of a composition using a kind of shifted Ferrers diagram [5]. Represent a part a_i as a row of a_i dots. Stack the rows so each starts where its predecessor ends. Then join up the dots with E and N steps.



stacked rows of dots for (2,1,2,3) — lattice path ENNENEE

If the compositions of n of a given length are listed in lex (dictionary) order, then so are the corresponding subsets, and the length of the conjugate is $n+1-$ length of the original. It follows that if the compositions of n are sorted, primarily by length and secondarily by lex order ($n = 4$ is shown),

$$(4),\ (1\ 3),\ (2\ 2),\ (3\ 1),\ (1\ 1\ 2),\ (1\ 2\ 1),\ (2\ 1\ 1),\ (1\ 1\ 1\ 1)$$

then the conjugate of the ith composition from the left is the ith composition from the right.

There are two statistics on compositions of n that are interchanged by conjugation, and these statistics again involve "singletons" (parts = 1) and "adjacent" parts. (For partitions, since they are unordered, adjacency necessarily means "in value" but here it means "in position"). To define them, observe that each part has two neighbors except the end parts which have only one or, in the case of a one-part composition, none ("wraparound" neighbors are not allowed here). Say a part ≥ 2 is big; a part $= 1$ is small.

Now define

$$\begin{aligned} \mu &= \text{sum of the big parts,} \\ \nu &= \text{sum of the small parts + total number of neighbors of the big parts.} \end{aligned}$$

For example, in the composition $(3,1,1,4,2)$, the big parts $3,4,2$ have $1,2,1$ neighbors respectively; so $\mu = 3+4+2 = 9$ and $\nu = (1+1)+(1+2+1) = 6$.

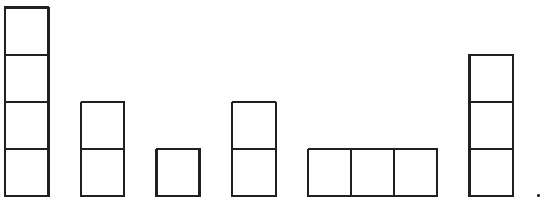
Theorem 2. *Conjugation interchanges the statistics μ and ν on compositions of n except when $n = 1$.*

Proof Carefully translate μ and ν to the corresponding lattice paths. Using the Iverson notation that $[statement] = 1$ if *statement* is true, $= 0$ if it is false, we find

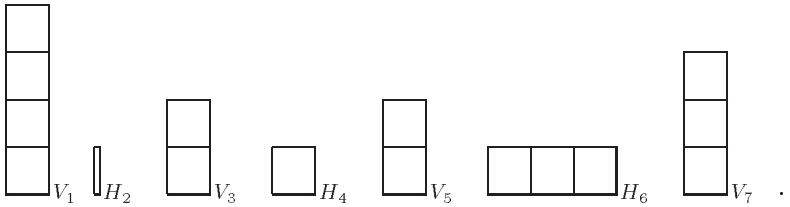
$$\begin{aligned} \mu &= \#E\text{s} + \#EN\text{s} + [\,\text{path ends } E\,], \text{ and} \\ \nu &= \#N\text{s} + \#EN\text{s} + [\,\text{path starts } N\,] \end{aligned}$$

It is then routine to check that μ on the flipped path agrees with ν on the original. □

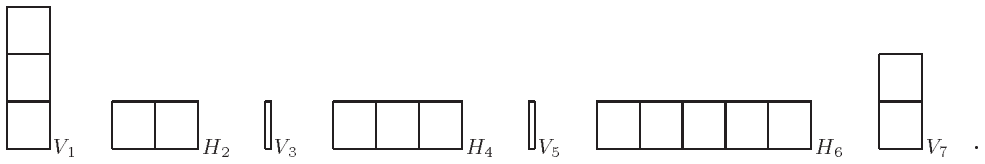
Finally, we remark that the genesis of the statistics μ and ν was the following graphical construction of the conjugate that involves "local" rather than "global" flipping. Suppose given a composition, say $(4,2,1,2,1^3,3)$, where consecutive 1s have been collected so that 1^3 is short for $1,1,1$. Represent it as a list of vertical strips (for parts > 1) and horizontal strips (for the 1s):



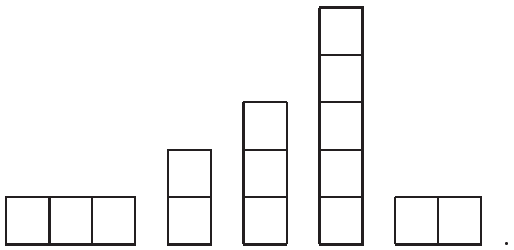
The vertical strips thus consist of 2 or more squares and may occur consecutively, but no two horizontal strips are consecutive. Insert an (initially) empty horizontal strip between each pair of consecutive vertical strips so that horizontal strips H and vertical strips V alternate (the first strip may be either an H or a V):



For each horizontal strip H, transfer one square from each of its neighboring vertical strips to H (there will be two such neighbors in general, but possibly just one or even none in the case of the all-1s composition):



Since each vertical strip originally contained ≥ 2 squares this will always be possible, though some vertical strips may afterward be empty, in which case just erase them. Finally, rotate all strips 90° so that Vs become Hs and vice versa:



The result is the conjugate composition: $(1^3, 2, 3, 5, 1^2)$.

References

[1] Robin Chapman, personal communication.

[2] Winston Yang, Bell numbers and k-trees, *Disc. Math.* **156** (1996), 247–252.

[3] Donald Knuth, Problem 11151, *Amer. Math. Monthly* **112**, April 2005, p. 367.

[4] Donald Knuth, *The Art of Computer Programming, Volume 4, Fascicle 3 : Generating All Combinations and Partitions*, Addison-Wesley Professional, 2005. Preliminary version .

[5] P. A. MacMahon, Combinatory Analysis, Vol. 1 *Cambridge University Press*, 1915; reprinted 2 vols. in 1, *Chelsea Publishing Company*, New York, 1984, p. 153.