

Another Determinant Condensation Formula

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It is shown that $CD = kDC$ implies that $k^r \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - kBC)$ where r is the total multiplicity of nonzero eigenvalues of C . This extends similar results known for the hypothesis $CD^T = \pm DC^T$, but its proof turns out to be more complicated. We also examine when our hypothesis is satisfied.

An Advanced Problem in the American Mathematical Monthly [1] prompted several exchanges on the use of continuity in establishing matrix identities. The idea is, you prove the identity for nonsingular matrices and then make an appeal to continuity to cover the singular case. M. J. Pelling gave a celebrated example [2] where this technique fails, for the very good reason that the result isn't true for singular matrices. In [3] Marvin Marcus showed that continuity is not in fact an issue in Pelling's example by giving a purely algebraic proof of essentially the following result (matrix entries in an integral domain).

THEOREM 1 Suppose A, B, C, D are n by n matrices satisfying

$$CD^T + DC^T = 0 \quad (1)$$

Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^r \det(AD^T + BC^T) \quad (2)$$

where $r = \text{rank}(C)$.

The point here is that the two determinants in (2) are necessarily equal when D is nonsingular but not necessarily equal when D is singular (assuming characteristic $\neq 2$, of course). To see this (and prove Theorem 1) note that (1) and (2) are unchanged when A, B, C, D are replaced by $Q^{-1}AQ, Q^{-1}B(Q^T)^{-1}, PCQ, PD(Q^T)^{-1}$ respectively, for any nonsingular matrices P, Q . Hence we may assume without loss of generality that $C = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ [the Smith normal form].

Let $\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ denote the corresponding block structure of D . Then (1) implies that $D_{21} = 0$ and $D_{11}^T = -D_{11}$. Simple determinant manipulations (as in the proof of our main result below, or see [4]) now serve to establish (2). Further, if D is nonsingular then $\det D_{11} \neq 0$ and so the size r of D_{11} is even (since $\det D_{11} = \det D_{11}^T = \det(-D_{11}) = (-1)^r \det D_{11}$) and the determinants in (2) are actually equal, as claimed.

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While this disposes of the continuity question, it does raise another algebraic question: can anything be said under the hypothesis $CD + DC = 0$? It turns out that an analog does indeed hold for this case. However its proof is a good deal more complicated, mainly because we are now restricted to replacing C —while leaving the problem invariant—by the Jordan form PCP^{-1} , rather than by the simpler Smith form PCQ . The proof of our main result, Theorem 2 below, follows some suggestions of Professor E. C. Dade which substantially improved and simplified the author's original proof.

THEOREM 2 *Let A, B, C, D be n by n matrices over an integral domain R with 1. Suppose*

$$CD = kDC \quad (3)$$

for some scalar k in R . Then

$$k^r \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - kBC) \quad (4)$$

where r is the total multiplicity of nonzero eigenvalues of C (and $0^0 := 1$).

Proof By embedding R in its field of quotients, we may assume that R is in fact a field. Since both hypothesis and conclusion are unchanged under conjugation of A, B, C, D by a suitable nonsingular matrix P , we may assume that C is a block diagonal matrix

$$C = \begin{pmatrix} C_{00} & 0 \\ 0 & C_{11} \end{pmatrix}$$

where C_{00} is nilpotent and C_{11} is nonsingular. Let n_0, n_1 denote the sizes of C_{00}, C_{11} respectively. We have $n_0, n_1 \geq 0, n_0 + n_1 = n$, and of course the number of nonzero eigenvalues is $r = n_1$. Say $\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}$ is the corresponding block structure of D .

Let us first dispose of the cases where $k = 0$. Case (i) $r (= n_1) > 0$. The hypothesis implies $D_{10} = 0$ and $D_{11} = 0$, whence D is singular and both sides of (4) vanish. Case (ii) $r (= n_1) = 0$. Here we must show ($0^0 := 1$)

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D \quad (5)$$

If $C_{00} (= C)$ is zero, (5) is trivial; if not, the hypothesis $CD = 0$ implies D is singular, hence has a left 0-eigenvector u . Choose i (≥ 0) maximal such that $uC^i \neq 0$ (possible since C is nilpotent). Then uC^i is a common left 0-eigenvector of C and D , whence the matrix $\begin{pmatrix} C & D \end{pmatrix}$ has deficient row rank and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is singular. Thus in this case both sides of (4) vanish. ■

Henceforth we assume $k \neq 0$.

LEMMA 1 $D_{10} = 0, D_{01} = 0$.

Proof The hypothesis implies $C_{00}D_{01} = kD_{01}C_{11}$ and, by iteration, $C_{00}^i D_{01} = k^i D_{01} C_{11}^i$. But C_{00} nilpotent implies $C_{00}^i = 0$ for some i , whence $D_{01} = 0$. Similarly $D_{10} = 0$.

Due to Lemma 1 we may conjugate further by an n by n matrix of the form $\begin{pmatrix} P_{00} & 0 \\ 0 & I \end{pmatrix}$ where P_{00} is n_0 by n_0 and nonsingular, to assume that

$$D_{00} = \begin{pmatrix} D_{00,00} & 0 \\ 0 & D_{00,11} \end{pmatrix}$$

where $D_{00,00}$ is n_{00} by n_{00} and nilpotent, while $D_{00,11}$ is n_{11} by n_{11} and nonsingular ($n_{00}, n_{11} \geq 0, n_{00} + n_{11} = n_0$). Let the corresponding block structure of C_{00} be

$$C_{00} = \begin{pmatrix} C_{00,00} & C_{00,01} \\ C_{00,10} & C_{00,11} \end{pmatrix}.$$

As in Lemma 1, we have $C_{00,01} = 0$ and $C_{00,10} = 0$, whence $C_{00,00}$ is nilpotent since C_{00} is. Thus

$$C = \begin{pmatrix} C_{00,00} & 0 & 0 \\ 0 & C_{00,11} & 0 \\ 0 & 0 & C_{11} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_{00,00} & 0 & 0 \\ 0 & D_{00,11} & 0 \\ 0 & 0 & D_{11} \end{pmatrix} \quad (6)$$

We complete the proof of (4) by considering two cases.

CASE (i) $n_{00} > 0$. From the hypothesis, we have $C_{00,00}^i D_{00,00}^j = k^{ij} D_{00,00}^j C_{00,00}^i$ as in Lemma 1, and then by an argument similar to that of the $k = 0$ case (ii) above, there exists a common left 0-eigenvector u_{00} and also a common right 0-eigenvector v_{00} of $C_{00,00}$ and $D_{00,00}$. Thus by (6),

$$u_0 := (u_{00}, 0) \left[\text{resp. } v_0 := \begin{pmatrix} v_{00} \\ 0 \end{pmatrix} \right]$$

is a common left [resp. right] 0-eigenvector of C and D . But then $(0, u_0) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0$, whence $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is singular and $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0$. Similarly $(AD - kBC)v_0 = 0$ and $\det(AD - kBC) = 0$. Thus in this case again both sides of (4) vanish.

CASE (ii) $n_{00} = 0$. This means we have

$$C = \begin{pmatrix} C_{00} & 0 \\ 0 & C_{11} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_{00} & 0 \\ 0 & D_{11} \end{pmatrix}$$

with C_{00} nilpotent and C_{11}, D_{00} both nonsingular. Determinant manipulations now suffice to establish (4). We have

$$\begin{aligned}
 \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det \begin{pmatrix} A_{00} & A_{01} & B_{00} & B_{01} \\ A_{10} & A_{11} & B_{10} & B_{11} \\ C_{00} & 0 & D_{00} & 0 \\ 0 & C_{11} & 0 & D_{11} \end{pmatrix} \\
 &= \det \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & A_{01} & B_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & A_{11} & B_{10} & B_{11} - A_{11}C_{11}^{-1}D_{11} \\ 0 & 0 & D_{00} & 0 \\ 0 & C_{11} & 0 & 0 \end{pmatrix} \\
 &= (-1)^{n_1} \det \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & B_{11} - A_{11}C_{11}^{-1}D_{11} \end{pmatrix} \det D_{00} \det C_{11} \quad (7)
 \end{aligned}$$

On the other hand, from the hypothesis $CD = kDC$ it follows directly that $D_{00}^{-1} \times C_{00}D_{00} = kC_{00}$ and $kC_{11}^{-1}D_{11}C_{11} = D_{11}$, and we have

$$\begin{aligned}
 \det(AD - kBC) &= \det \begin{pmatrix} A_{00}D_{00} - kB_{00}C_{00} & A_{01}D_{11} - kB_{01}C_{11} \\ A_{10}D_{00} - kB_{10}C_{00} & A_{11}D_{11} - kB_{11}C_{11} \end{pmatrix} \\
 &= \det \begin{pmatrix} A_{00}D_{00} - B_{00}D_{00}^{-1}C_{00}D_{00} & kA_{01}C_{11}^{-1}D_{11}C_{11} - kB_{01}C_{11} \\ A_{10}D_{00} - B_{10}D_{00}^{-1}C_{00}D_{00} & kA_{11}C_{11}^{-1}D_{11}C_{11} - kB_{11}C_{11} \end{pmatrix} \\
 &= (-k)^{n_1} \det \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & B_{11} - A_{11}C_{11}^{-1}D_{11} \end{pmatrix} \det D_{00} \det C_{11} \quad (8)
 \end{aligned}$$

Since $r = n_1$, (4) follows immediately from (7) and (8). This concludes the proof of Theorem 2. $\blacksquare$

Of course, the matrix C does not have a privileged role over that of D . In fact, Theorem 2 also holds defining r to be the total multiplicity of zero eigenvalues of D . The proof is much like that of Theorem 2 when $k = 0$; and for $k \neq 0$ simply switch the block columns in $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, apply Theorem 2 to the resulting matrix, then switch back.

Now we may ask: do any nontrivial matrices satisfy $CD = kDC$. The equation $AX = XB$ is fully solved over $\mathbb{C}$ in [5, p. 421] and the results imply the following Theorem (for $F = \mathbb{C}$).

THEOREM 3 Suppose $C \in F^{n \times n}$ (n by n matrices over F) and let $\{p_i(x)^{e_i}\}_{i=1}^t$ denote the elementary divisors of C .

Then the dimension of the vector space $V := \{D \in F^{n \times n} : CD = kDC\}$ is given by

$$\begin{aligned} \dim_F V &= \sum_{i=1}^t \sum_{j=1}^t \deg(\text{g.c.d.}\{p_i(kx)^{e_i}, p_j(x)^{e_j}\}) & k \neq 0 \\ &= n \sum_{i=1}^t \deg(\text{g.c.d.}\{x, p_i(x)^{e_i}\}) & k = 0 \end{aligned}$$

(Thus nontrivial D exist iff some pair of eigenvalues of C have ratio k .)

Since the spirit of the present paper is algebraic generality, we include a proof that Theorem 3 is in fact true for arbitrary fields F .

LEMMA Suppose A, B are square matrices (not necessarily the same size) with entries in F . Then $AX = XB$ has only the trivial solution for X unless A, B have a common eigenvalue (in $\bar{F}$).

Proof One method is to realize the matrix equation $AX = XB$ as a homogeneous linear system of equations for the entries in X with coefficient matrix $A \times I - I \times B^T$ (Kronecker products) whose eigenvalues are known to be $\lambda_A - \lambda_B$ where λ_A ranges over the eigenvalues of A and similarly for λ_B . Another method is to use Jordan canonical forms for A and B (permissible since $AX = XB \Leftrightarrow (PAP^{-1})(PXQ) = (PXQ)(Q^{-1}BQ)$). This reduces the problem to solving $(\lambda_A I + J_1)X = X(\lambda_B I + J_2)$ with J_1, J_2 nilpotent, which is quite easy. The details are left to the reader. ■

Proof of Theorem 3 We treat the case $k \neq 0$ (leaving $k = 0$ as an exercise). We may clearly assume by conjugation that C is in primary rational canonical form $C = \text{diag}(C_1, C_2, \dots, C_t)$ where C_i has minimum polynomial $p_i(x)^{e_i}$, p_i irreducible over F . Partitioning D into corresponding blocks D_{ij} , the hypothesis $CD = kDC$ is equivalent to $C_i D_{ij} = k D_{ij} C_j$ for all i, j . We count the contribution of each of these equations to $\dim V$. Now $\deg(\text{g.c.d.}\{p_i(kx)^{e_i}, p_j(x)^{e_j}\}) = 0$ iff $\text{g.c.d.}(p_i(kx), p_j(x)) = 1$ iff C_i and kC_j have no common eigenvalue, and this implies $D_{ij} = 0$ by the lemma. On the other hand $\deg(\text{g.c.d.}\{p_i(kx)^{e_i}, p_j(x)^{e_j}\}) > 0$ iff $p_i(kx) = k^{\deg p_j} p_j(x)$ (since p_i, p_j are irreducible), from which it follows that the minimum polynomial of kC_j is $p_i(x)^{e_j}$, and $\deg(\text{g.c.d.}\{p_i(kx)^{e_i}, p_j(x)^{e_j}\}) = \deg p_i^{\min\{e_i, e_j\}}$. Thus, by conjugating, we may assume [6, p. 203 Ex. 13] that $C_i = P_i$ and $kC_j = P_j$ where P_m denotes the matrix

$$\begin{pmatrix} P & N & & 0 \\ & P & \ddots & \\ 0 & & \ddots & \ddots & N \\ & & & \ddots & P \end{pmatrix}$$

[with m P 's], P is the companion matrix of $p_i(x)$ and N 's only nonzero entry is a 1 in the upper right corner.

If $e_i = e_j$, this means D_{ij} commutes with the "cyclic" matrix C_i , hence by a basic theorem, D_{ij} is a polynomial in C_i so " D_{ij} " contributes $\deg p_i^{e_i}$ to $\dim V$, just as desired. If $e_i \neq e_j$, enlarge D_{ij} by adding zeros to produce a square matrix $Y = \begin{pmatrix} D_{ij} \\ 0 \end{pmatrix}$ or $(0 \ D_{ij})$. Letting $e = \max(e_i, e_j)$, we now find $P_e Y = Y P_e$. Hence Y is a

polynomial in P_e and so Y 's blocks (as induced by P_e) necessarily form an upper triangular Toeplitz matrix (equal entries along each diagonal). This readily yields $D_{ij} = (0 \ E)$ or $\begin{pmatrix} E \\ 0 \end{pmatrix}$, where E is square of size $m = \min\{e_i, e_j\}$. Again, $P_m E = E P_m$ whence E is a polynomial in P_m : thus here " D_{ij} " contributes $\deg p_i^{\min\{e_i, e_j\}}$ to $\dim V$, and we are done.

Finally, the transpose case of our main result. It is true, by way of analogy, that $CD^T = kDC^T$ implies

$$k^{\text{rank}(C)} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD^T - kBC^T)$$

($0^0 := 1$) but this is not a significant generalization. For the case $k = 1$ is proved by Dade in [2], $k = -1$ is established above, (they could be handled together, of course), $k = 0$ is a straightforward exercise and no other cases are possible. This is easily seen since $(DC^T)^T = CD^T$, and the equation $M^T = kM$ with $M \neq 0$ immediately yields $k^2 = 1$.

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References

- [1] Problem 6057, *American Mathematical Monthly* **84** (June–July, 1977), 495–496.
- [2] Postmortem Comment, *American Mathematical Monthly* **94** (December 1987), 1020–1021.
- [3] M. Marcus, Two determinant condensation formulas, *Linear and Multilinear Algebra* **22** (1987), 95–102.
- [4] D. Callan, Solution of E3286, proposed by W. C. Waterhouse, *American Mathematical Monthly* **97** (March 1990), 249–250.
- [5] P. Lancaster and M. Tismenetzky, *The Theory of Matrices*, 2nd ed., Academic Press, 1985.
- [6] N. Jacobson, *Basic Algebra I*, 2nd ed., W. H. Freeman and Co., 1985.