Some Bijections for Restricted Motzkin Paths

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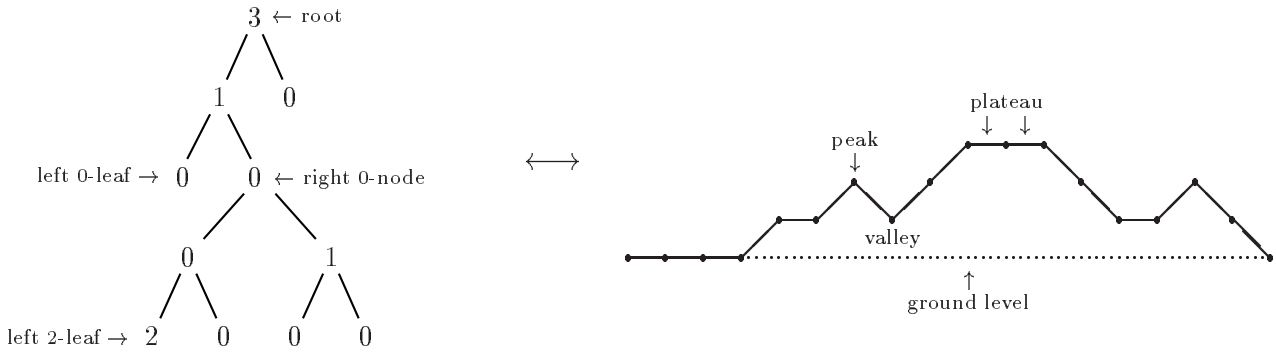
July 19 2004

There is a classic correspondence between full binary trees on $2n$ edges and Dyck paths of $2n$ steps [1, Ex. 6.19d, 6.19i]: traverse the tree in preorder (counterclockwise from the root) and, as each edge is encountered for the first time, record an upstep for a left edge and a downstep for a right edge. This correspondence readily extends to Motzkin paths because a Motzkin path can be viewed as a Dyck path with a nonnegative label on each vertex recording the number of flatsteps at that location, and Dyck path vertices in left-to-right order correspond to tree vertices in preorder. So just label each vertex in the tree with the number of flatsteps at the corresponding location in the path as illustrated below.



Motzkin tree Motzkin path

Motzkin tree-path correspondence

The weight of the labeled tree is # edges + sum of labels, and the length of a path is its number of steps. So weight of tree $\leftrightarrow$ length of path. We will call such a labeled tree of weight n a Motzkin n-tree and a Motzkin path of length n a Motzkin n-path so that

Motzkin n-trees correspond to Motzkin n-paths. (Motzkin n-trees are closely related to the $\{0,1,2\}$-trees of [2].) We mostly follow the notation in [2], in particular distinguishing between a node (non-root interior vertex) and a leaf. Thus the vertices of a Motzkin tree are partitioned into a root, a set of nodes, and a set of leaves. Each node and leaf is left or right according as it is a left or right child of its parent. A k-node is one whose label is k and a positive node is one whose label is ≥ 1. Similarly for leaves. The trivial tree has no leaves. In a nontrivial tree, the first and last leaf are as encountered in preorder, so the first leaf is left and the last is right. The level of a vertex is the length of the unique path joining it to the root. Every non-root vertex has a unique sibling—the other child of its parent. We use U for upstep, F for flatstep, and D for downstep. A plateau in a path is a run of flatsteps that is either the entire path or of length ≥ 1 and preceded by an upstep and followed by a downstep.

We recall some obvious correspondences.

Motzkin tree	**Motzkin path**
root label	# initial Fs
first leaf	first peak or plateau
last leaf	terminal vertex
left 0-node	doublerise (UU)
right 0-node	valley (DU)
left 0-leaf	peak (UD)
right 0-leaf (except last leaf)	doublefall (DD)
level of first leaf	height of first peak or plateau
level of last leaf	# Ds that return path to ground level
positive labels on left leaves	plateau lengths

We use $\mathcal{M}_n(UD, DU)$ to denote the set of Motzkin n-paths that contain neither peaks nor valleys, and so on. Thus, for example $\mathcal{M}_n(UU)$ corresponds to the set of Motzkin n-trees in which each left node is positive.

Each of the following 5 bijections has both a recursive and an explicit description. The recursive specification depends on the first few steps and the first return to ground level; ε denotes the empty path, R, S, T denote Motzkin paths. Motzkin trees facilitate the explicit description. The equivalence of the recursive and explicit descriptions can be

proved by induction. Emeric Deutsch [3] found the recursive form of most of them.

Bijection 1. $\phi : \mathcal{M}_n \to \mathcal{M}_n$.

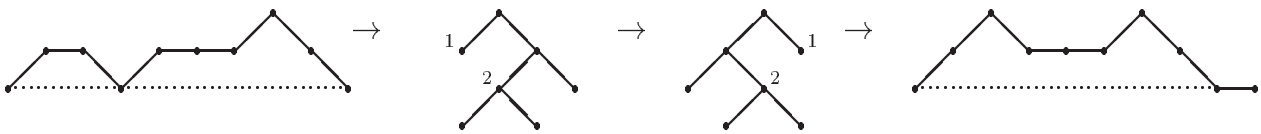
Recursive:

$$\begin{aligned} \phi(\varepsilon) &= \varepsilon \\ \phi(FR) &= F\phi(R) \\ \phi(URDS) &= U\phi(S)D\phi(R) \end{aligned}$$

Explicit:

$$\text{path} \xrightarrow[\text{above}]{\text{as}} \text{tree} \xrightarrow[\text{vertical}]{\text{flip in}} \text{tree} \xrightarrow[\text{above}]{\text{as}} \text{path}$$

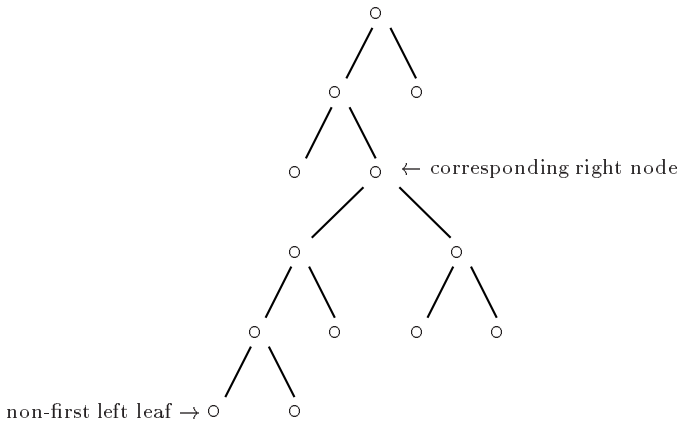
Example: (0 labels omitted)



Consequence: Since left 0-nodes (UU) and right 0-nodes (DU) are exchanged after the flip, the parameters $\#UU$s and $\#DU$s have the same distribution on $\mathcal{M}_n$. In particular, $|\mathcal{M}_n(UU)| = |\mathcal{M}_n(DU)|$ (A004148).

Remark: This bijection is clearly an involution on $\mathcal{M}_n$ and generalizes one on Dyck paths [4]. □

In a full binary tree there is an obvious correspondence between non-first left leaves and right nodes: given such a leaf, travel (toward the root) to the first right node encountered.



Note that this correspondence holds (vacuously) even for the trivial tree consisting of the root alone. By associating the root to the first left leaf, we can extend this correspondence to {left leaves} $\leftrightarrow$ {right nodes} $\cup$ {root}, except in the case of the trivial tree. This exception ultimately accounts for why many of our bijections need to increase the path length (tree weight) by 1.

Similarly, {right leaves} $\leftrightarrow$ {left nodes} $\cup$ {root} in all but the trivial tree. Applied to a Dyck path (all labels 0), these correspondences yield the obvious fact that #peaks = #valleys +1 and the slightly less obvious fact that #doublerises = #doublefalls.

The next 3 bijections are all from $\mathcal{M}_n(UU)$ to $\mathcal{M}_{n+1}(UD)$.

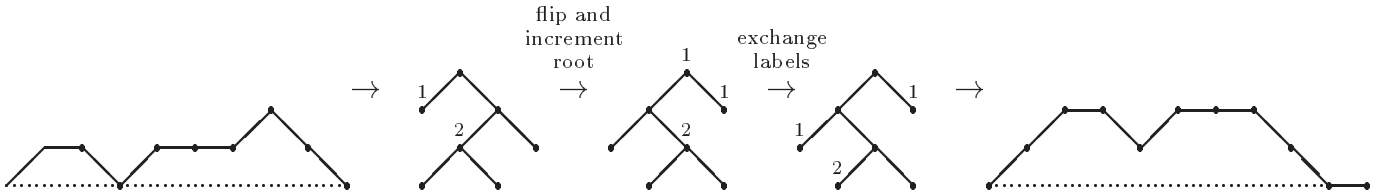
Bijection 2. $\phi : \mathcal{M}_n(UU) \to \mathcal{M}_{n+1}(UD)$.

Recursive:

$$\begin{aligned}
\phi(\varepsilon) &= F \\
\phi(FR) &= F\phi(R) \\
\phi(UF^aDR) &= U\phi(R)DF^a \qquad\qquad a \geq 0 \\
\phi(UF^aRF^bDS) &= U\phi(S)D\phi(F^bRF^{a-1}) \quad a \geq 1,\ b \geq 0;\ R \text{ starts } U, \text{ ends } D
\end{aligned}$$

Explicit: Given a UU-free Motzkin n-path, its tree has positive labels on its left nodes. Flip the tree in the vertical and increment the root label by 1. Exchange labels on nonfirst left leaves and corresponding right nodes, and exchange the labels on the first leaf and the root. Now every left leaf has a positive label and the tree weight is incremented by 1. Take the corresponding path—a UD-free Motzkin $(n+1)$-path. The map is obviously reversible.

Example:



Consequence: The parameters $\#DD$s on $\mathcal{M}_n(UU)$ and $\#DU$s on $\mathcal{M}_{n+1}(UD)$ have the same distribution. This is because, after the tree flip and label exchange, a non-last right 0-leaf (DD) becomes a right 0-node (DU). In particular, $|\mathcal{M}_n(UU, DD)| =$

$|\mathcal{M}_{n+1}(UD, DU)|$ or, more picturesquely, Motzkin n-paths containing no long slanted segments are equinumerous with Motzkin $(n+1)$-paths containing no sharp turns (A004149). □

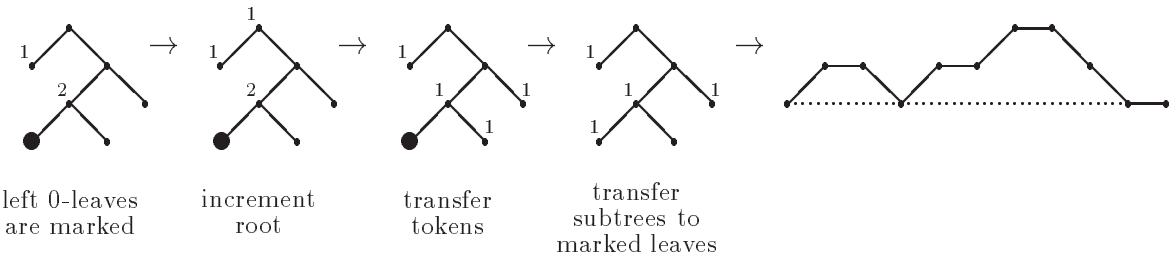
Bijection 3. $\phi : \mathcal{M}_n(UU) \to \mathcal{M}_{n+1}(UD)$.

Recursive:

$$\begin{aligned} \phi(\varepsilon) &= F \\ \phi(FR) &= F\phi(R) \\ \phi(UDR) &= U\phi(R)D \\ \phi(UFRDS) &= U\phi(R)D\phi(S) \end{aligned}$$

Explicit: Given a UU-free Motzkin n-path, its tree has positive labels on its left nodes. Consider the labels as counting tokens (flatsteps) stored at their location.
Step 1. Add a token to the root.
Step 2. Transfer one token from each left node and from the root to its corresponding right leaf. (Except do nothing if the tree consists of the root alone.)
Step 3. For each left 0-leaf, transfer the subtree of its sibling vertex (including the label on the sibling vertex) to this left leaf. Note that after the transfer, the left 0-leaf may become a node or a positive leaf but will no longer be a 0-leaf. Also, the sibling in question becomes a right 0-leaf and all other right leaves are positive due to Step 2. The weight has been increased by 1 and all left leaves are now positive, so the resulting path is indeed in $\mathcal{M}_{n+1}(UD)$. The map is reversible: the original left 0-leaves are recovered as the siblings of right 0-leaves in the image.

Example: (same UU-free path as in the previous example)



Consequence: Define a low peak in a Motzkin path to be a peak whose downstep returns the path to ground level, and the final descent to be the one that terminates the

path (assumed empty if the path ends with a flatstep). Then the parameters "#low peaks" on $\mathcal{M}_n(UU)$ and "length of final descent" on $\mathcal{M}_{n+1}(UD)$ have the same distribution. In particular, $|\{P \in \mathcal{M}_n(UU) : P \text{ has no low peaks}\}| = |\mathcal{M}_n(UD)|$. This is because the image ends F (i.e. has final descent 0) $\Leftrightarrow$ the original path has no low peaks. Deleting this final F is a bijection to $\mathcal{M}_n(UD)$. □

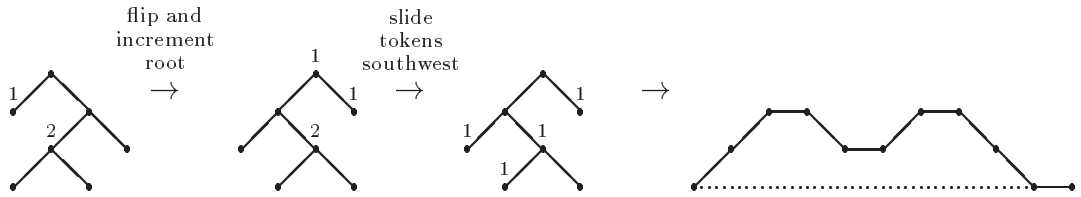
Bijection 4. $\phi : \mathcal{M}_n(UU) \to \mathcal{M}_{n+1}(UD)$.

Recursive:

$$\begin{aligned} \phi(\varepsilon) &= F \\ \phi(FR) &= F\phi(R) \\ \phi(UDR) &= U\phi(R)D \\ \phi(UFRDS) &= U\phi(S)D\phi(R) \end{aligned}$$

Explicit: Given a UU-free Motzkin n-path, pass to its corresponding tree, flip tree in vertical and add a token to the root. Now the root and right nodes all have positive labels. Transfer one token from each to the corresponding left leaf (do nothing if there are no edges). Here again, all left leaves are now positive, giving a UD-free Motzkin $(n+1)$-path.

Example: (same UU-free path as in the previous two examples)



Consequence: The parameters $\#UFU$s on $\mathcal{M}_n(UU)$ and $\#DU$s on $\mathcal{M}_{n+1}(UD)$ have the same distribution. This is because a UFU occurs for each left node with label 1. After the flip, it becomes a right node with label 1. The required transfer of tokens makes it a right 0-node, and a DU in the image path. □

Next, we consider a bijection that increases plateau lengths in valley-free paths. Recall that a $U\underbrace{F\ldots F}_{k\geq 1}D$ sequence is a plateau of length k (as is $\underbrace{F\ldots F}_{k\geq 0}$ if it is the entire path).

Let MPL denote minimum plateau length (A064645) in a path (taken to be 0 if there are no plateaus).

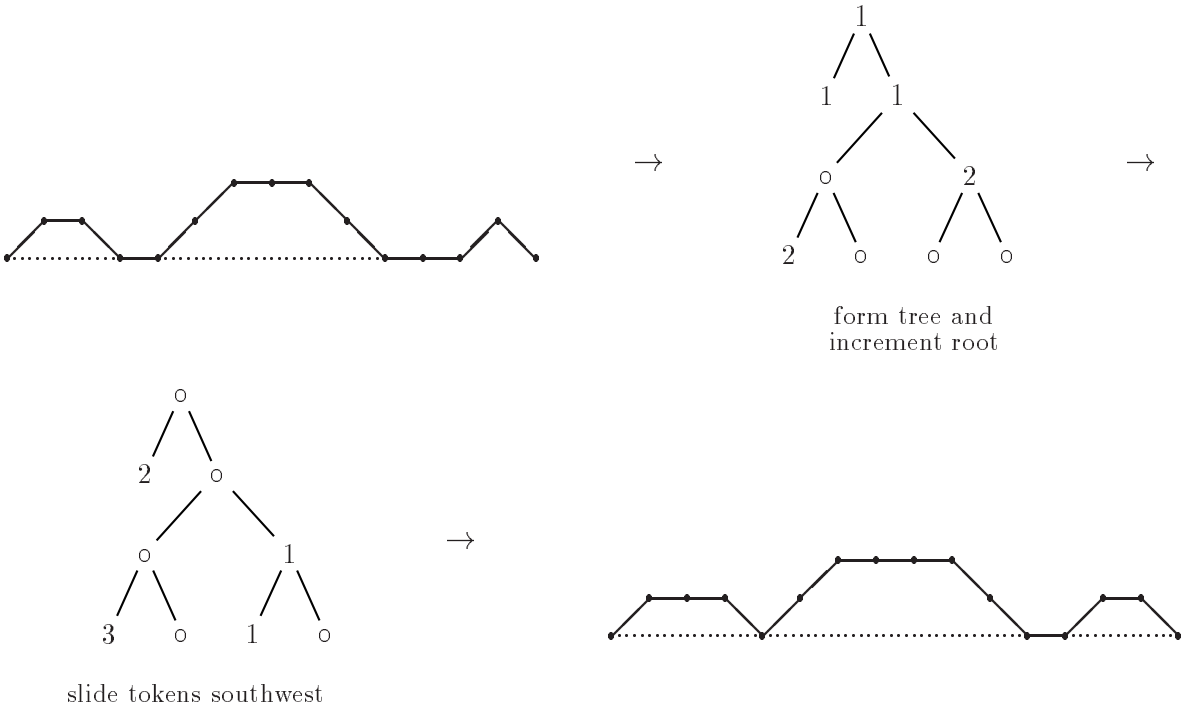
Bijection 5. $\phi : \mathcal{M}_n(DU) \to \mathcal{M}_{n+1}(UD)$.

Recursive:

$$\begin{aligned} \phi(\varepsilon) &= F \\ \phi(FR) &= F\phi(R) \\ \phi(URD) &= U\phi(R)D \\ \phi(URDFS) &= U\phi(R)D\phi(S) \end{aligned}$$

Explicit: A DU-free path gives a tree in which all right nodes have a positive label. Add a token to the root and (unless it's the root-only tree) transfer a token from the root and from each right node to its corresponding left leaf. Now all left leaves have a positive label, giving a UD-free Motzkin $(n+1)$-path. The map is clearly reversible.

Example:



Consequence: The parameters $MPL + 1$ on $\mathcal{M}_n(DU)$ and MPL on $\mathcal{M}_{n+1}(UD)$ have the same distribution. This follows since each peak (left 0-leaf) becomes a plateau of length 1 and each existing plateau has its length increased by 1. □

Bijection 4 followed by the inverse of Bijection 2, that is, $\phi_2^{-1}\phi_4$, shows that the parameters $\#UFU$s and $\#DD$s are equidistributed on $\mathcal{M}_n(UU)$. In fact, this bijection is an involution on $\mathcal{M}_n(UU)$ that interchanges occurrences of UFU and DD. We conclude with a simple explicit description of $\phi_2^{-1}\phi_4$. For this purpose say an upstep is critical if it is followed by an F. A strict Motzkin path is one that starts U and ends D. Given a UU-free path, if the path segment strictly between a critical U and its matching D is level (all Fs), leave it alone. Otherwise, it has the form F^aSF^b with $a \geq 1$, $b \geq 0$, S strict. Replace it by F^bSF^{a-1}. The result is independent of the order in which critical Us are processed and is again UU-free. This map is clearly an involution and, because of the restriction to UU-free paths, DDs in the image path correspond one-to-one to UFUs in the original.

Acknowledgement

I thank Emeric Deutsch for several helpful comments on an early draft of this paper.

References

[1] Richard P. Stanley, *Enumerative Combinatorics Vol. 2*, Cambridge University Press, 1999. Exercise 6.19 is available online as Catalan Addendum.

[2] Emeric Deutsch, Louis W.Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, *Discrete Math.* **256** (2002), 655-670.

[3] Emeric Deutsch, unpublished notes.

[4] Emeric Deutsch, An involution on Dyck paths and its consequences, *Discrete Math.* **204** (1999), 163-166.