

# Flexagons Yield a Curious Catalan Number Identity

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**Dedicated to the memory of Martin Gardner, 1914–2010.**

## Abstract

Hexaflexagons were popularized by the late Martin Gardner in his first Scientific American column in 1956. Oakley and Wisner showed that they can be represented abstractly by certain recursively defined permutations called pats, and deduced that they are counted by the Catalan numbers. Counting pats by number of descents yields the curious identity

$$\sum_{k=0}^n \frac{1}{2n-2k+1} \binom{2n-2k+1}{k} \binom{2k}{n-k} = C_n,$$

where only the middle third of the summands are nonzero.

## 1 Introduction

Martin Gardner [1] showed how to construct **hexaflexagons** in his 1956 debut column in Scientific American. Soon after, a mathematical treatment by Oakley and Wisner appeared in the Monthly [2]. They identified hexaflexagons with certain integer permutations that they called pats (I don’t know why). Pats are defined recursively, with permutations represented as lists. A singleton permutation is a pat, and a permutation  $p$  of length  $n \geq 2$  is a pat iff (i) there is a unique split point that divides  $p$  into subpermutations  $p_1, p_2$  such that all entries in  $p_1$  are greater than all entries in  $p_2$ , and (ii) the reverse of each of  $p_1$  and  $p_2$  is a pat. The pats on [3] are 231 and 312, and on [4] are 2431, 3241, 3412, 4132, 4213. The number of pats on  $[n+1]$  (as Oakley and Wisner showed and will be evident) is the Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ .

The purpose of this note is to find the distribution of the statistic “number of descents” on pats. (A descent in a permutation  $p$  is a pair of adjacent entries  $(p_i, p_{i+1})$  with  $p_i > p_{i+1}$ .) To do so, we establish a recurrence that leads to a quartic equation for the

generating function. Lagrange inversion then yields a simple closed form for the number of pats on  $[n]$  with  $k$  descents, thereby giving a combinatorial interpretation of the identity

$$\sum_{k=0}^n \frac{1}{2n-2k+1} \binom{2n-2k+1}{k} \binom{2k}{n-k} = C_n. \quad (1)$$

The last section exhibits a bijection from pats to full binary trees showing that descents on pats are distributed as even-level vertices on binary trees and concludes with a generalization of (1).

## 2 Distribution of Descents on Pats

From its definition, a pat  $p$  of length  $n$  determines two pats  $p_1, p_2$  of lengths  $i$  and  $n-i$  respectively for some  $i \in [1, n-1]$ . The connecting entries from  $p_1$  to  $p_2$  contribute a descent to  $p$  and the remaining descents of  $p$  correspond to the ascents in  $p_1$  and  $p_2$ . Hence we obtain the recurrence

$$u(n, k) = \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} u(i, j) u(n-i, n-j-k-1) \quad n \geq 2, k \geq 0$$

for the number  $u(n, k)$  of pats on  $[n]$  with  $k$  descents, with  $u(1, 0) = 1$ . The recurrence leads directly to a functional equation for the generating function  $F(x, y) := \sum_{n \geq 1, k \geq 0} u(n, k) x^n y^k$ :

$$F(x, y) = x + \frac{1}{y} F(xy, \frac{1}{y})^2. \quad (2)$$

Iterating (2) once yields the algebraic equation

$$F = x + y(x + F^2)^2,$$

where  $F(x, y)$  is now abbreviated to  $F$ . Direct application of Lagrange inversion to solve this equation is cumbersome but a trick shown to me by Ira Gessel (see [3]) rapidly solves it. Introduce a new variable  $z$  and consider  $F = F(x, y, z)$  defined by

$$F = z(x + y(x + F^2)^2) \quad (3)$$

Equation (3) has the form  $z = F/\phi(F)$  where

$$\phi(F) = x + y(x + F^2)^2,$$

and so Lagrange inversion says that

$$[y^k z^{2j+1}] F(x, y, z) = \frac{1}{2j+1} [y^k F^{2j}] \phi(F)^{2j+1} = \frac{1}{2j+1} \binom{2j+1}{k} \binom{2k}{j} x^{j+k+1}. \quad (4)$$

The coefficient of  $x^n y^k$  in  $F(x, y, 1)$  is obtained by setting  $j = n - k - 1$  in (4), yielding

$$u(n, k) = [x^n y^k] F(x, y) = [x^n y^k] F(x, y, 1) = \frac{1}{2n - 2k - 1} \binom{2n - 2k - 1}{k} \binom{2k}{n - k - 1},$$

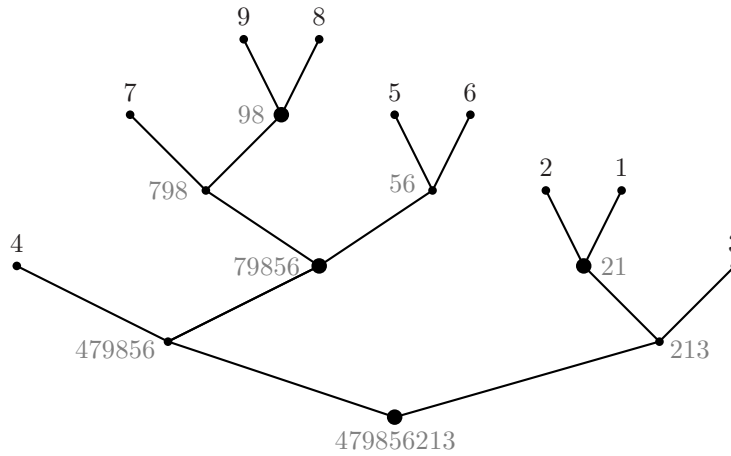
and hence identity (1) after replacing  $n$  by  $n + 1$ . Here are the first few values of  $u(n, k)$ .

| $n \setminus k$ | 0 | 1 | 2  | 3   | 4    | 5    | 6   |
|-----------------|---|---|----|-----|------|------|-----|
| 1               | 1 | 0 | 0  | 0   | 0    | 0    | 0   |
| 2               | 0 | 1 | 0  | 0   | 0    | 0    | 0   |
| 3               | 0 | 2 | 0  | 0   | 0    | 0    | 0   |
| 4               | 0 | 1 | 4  | 0   | 0    | 0    | 0   |
| 5               | 0 | 0 | 12 | 2   | 0    | 0    | 0   |
| 6               | 0 | 0 | 12 | 30  | 0    | 0    | 0   |
| 7               | 0 | 0 | 4  | 100 | 28   | 0    | 0   |
| 8               | 0 | 0 | 0  | 140 | 280  | 9    | 0   |
| 9               | 0 | 0 | 0  | 90  | 980  | 360  | 0   |
| 10              | 0 | 0 | 0  | 22  | 1680 | 2940 | 220 |

Table of values of  $u(n, k)$ , the distribution of descents on pats

### 3 Pats as Trees

A pat on  $[n + 1]$  can be represented, using its successive split points, by a vertex-labeled full binary tree on  $2n$  edges as illustrated below.



full binary tree for pat 479856213,  
interior vertices at even level are enlarged

In fact, the labels are unnecessary; they can be uniquely recovered from the underlying tree. So we have a bijection from pats on  $[n + 1]$  to full binary trees on  $2n$  edges. Under this bijection, a descent in the pat shows up as an interior vertex at even level in the tree (where rainwater would collect between the two leaves corresponding to the descent).

Thus we have, pruning the leaf edges in a full binary tree to obtain a binary tree,

**Theorem** *Descents on pats are distributed as even-level vertices in binary trees.*

Similar considerations for ternary and higher order trees yield a generalization of (1):

$$\sum_{k=0}^n \frac{1}{rn - rk + 1} \binom{rn - rk + 1}{k} \binom{rk}{n - k} = \frac{1}{rn + 1} \binom{rn + 1}{n}$$

for  $r \geq 2$ .

**Acknowledgment** I thank Ira Gessel for suggesting the Lagrange inversion technique used in Section 2.

## References

- [1] Martin Gardner, *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games*, University of Chicago Press, 1988.
- [2] C.O. Oakley and R.J. Wisner, Flexagons, *Amer. Math. Monthly* **64**, March 1957, 143–154.
- [3] Ira Gessel, A combinatorial proof of the multivariable Lagrange inversion formula. *J. Combin. Theory Ser. A* **45**, 1987, 178–195.