

GAUSSIAN INEQUALITY

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ABSTRACT. We prove some special cases of Bergeron's inequality involving two Gaussian polynomials (or q -binomials).

1. INTRODUCTION

We begin by recalling the q -analogues $[n]!_q = \prod_{j=1}^n \frac{1-q^j}{1-q}$ of factorials and the q -analogue $\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$ of binomial coefficients. Adopt the convention $[0]!_q = 1$. It is well-known that these rational functions $\binom{n}{k}_q$ are polynomials, in q , also called *Gaussian polynomials*, having non-negative coefficients which are also *unimodal* and symmetric. Furthermore, there are several combinatorial interpretations of which we state two of them.

A *word* of length n over the *alphabet* set $\{0, 1\}$ is a finite sequence $w = a_1 \cdots a_n$. Construct

$$\mathcal{W}_{n,k} = \{w = a_1 \cdots a_n : w \text{ has } k \text{ zeros and } n - k \text{ ones}\}$$

and the *inversion set* of w as $\text{Inv}(w) = \{(i, j) : i < j \text{ and } a_i > a_j\}$. The corresponding *inversion number* of w will be denoted $\text{inv}(w) = \#\text{Inv}(w)$. Then, we have

$$\binom{n}{k}_q = \sum_{w \in \mathcal{W}_{n,k}} q^{\text{inv}(w)}.$$

Yet, another formulation which would come to appeal to many combinatorialists is

$$\binom{a+d}{a}_q = \sum_T q^{\text{area}(T)}$$

where T is a lattice path inside an $a \times d$ box and $\text{area}(T)$ is area above the curve T .

Given two polynomials $f(q)$ and $g(q)$, we write $f(q) \geq g(q)$ provided that $f(q) - g(q)$ has non-negative coefficients in the powers of q .

The well-known *Foulkes conjecture* (see, for instance [4]) was generalized by Vessenes [8]. She conjectured that

$$(h_b \circ h_c) - (h_a \circ h_d) \tag{1}$$

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is *Schur positive* (expands with positive integer coefficients in the Schur basis $\{s_\mu\}_{\mu \vdash n}$ of symmetric polynomials) whenever $a \leq b \leq c \leq d$, with $n = ad = bc$, and one writes $(h_n \circ h_k)$ for the *plethysm* of complete homogeneous symmetric functions. A well-known fact is that $(h_n \circ h_k)(1, q) = \binom{n+k}{k}_q$. Moreover, any non-zero evaluation of a Schur function at 1 and q is of the form $q^i + q^{i+1} + \dots + q^j$ for some $i < j$. Exploiting these facts on the occasion of [3], and assuming that (1) holds, F. Bergeron (see also [4]) underlined that the evaluation of the difference in (1), at 1 and q , would imply the following:

Conjecture 1. *Assume $0 < a \leq b \leq c \leq d$ are positive integers with $ad = bc$. Then, the following difference of two Gaussian polynomials is symmetric and satisfies*

$$\binom{b+c}{b}_q - \binom{a+d}{a}_q \geq 0. \quad (2)$$

One can associate a direct combinatorial meaning to Vessenenes' conjecture in the context of representation theory of $GL(V)$. Indeed, if it holds true, it would signify that there is an embedding of the composite of symmetric powers $S^a(S^d(V))$ inside $S^b(S^c(V))$, as $GL(V)$ -modules. It may however be more natural to state that there is a surjective $GL(V)$ -module morphism the other way around (which is also equivalent). Therefore each $GL(V)$ -irreducible occurs with smaller multiplicity in $S^a(S^d(V))$ than it does in $S^b(S^c(V))$, and the conjecture reflects this at the level of the corresponding characters (with Schur polynomials appearing as characters of irreducible representations).

The sole attempt [9] toward resolving Conjecture 1 was made by F. Zanello, who attends to the special case $a \leq 3$, including the property of symmetry and unimodality. A sequence of numbers is *unimodal* if it does not increase strictly after a strict decrease. The author in [9] offers a strengthening of Conjecture 1 to the effect that

Conjecture 2. *Preserve the hypothesis in Conjecture 1. Then, the coefficients of the symmetric polynomial*

$$\binom{b+c}{b}_q - \binom{a+d}{a}_q$$

are non-negative and unimodal.

Notice that symmetry is clear, since both $\binom{b+c}{b}_q$ and $\binom{a+d}{a}_q$ are symmetric polynomials of the same degree, $ad = bc$. We started out this project with the goal of proving the below 3-parameter special case of Conjecture 1 which we dubbed the β -Conjecture. Namely,

Conjecture 3. *For integers $0 < a < b$ and $\beta \geq 1$, we have*

$$\binom{b+\beta a}{b}_q \geq \binom{a+\beta b}{a}_q. \quad (3)$$

The case $\beta = 1$ is trivial. However, our journey in this effort failed short of capturing the β -Conjecture in its fullest. In the sequel, we supply the details of our success in settling the particular instance $\beta = 2$. Let's commence by stating one useful identity.

Theorem 1. (*q-analogue Vandermonde-Chu*). *The following holds true*

$$\sum_{j \geq 0} \binom{X}{Z-j}_q \binom{Y}{j}_q q^{j(X-Z+j)} = \binom{X+Y}{Z}_q. \quad (4)$$

Remark 2. In view of (4), Conjecture 1 tantamount

$$\binom{b+c}{b}_q = \sum_{k=0}^b \binom{b}{k}_q \binom{c}{k}_q q^{k^2} \geq \sum_{k=0}^a \binom{a}{k}_q \binom{d}{k}_q q^{k^2} = \binom{a+d}{a}_q.$$

2. THE CASE $\beta = 2$ AND $q = 1$

In this section, we wish to explain the resolution of the β -Conjecture for the ordinary binomial coefficients ($q = 1$) while $\beta = 2$, which elaborates a natural development.

For $a < b < c < d$ with $ad = bc$ and special case $c = 2a, d = 2b$, say $b = a + i, i \geq 1$, it would be desirable to find a bijective proof for

$$\binom{3a+i}{a+i} \geq \binom{3a+2i}{a}.$$

An injection from a set counted by the smaller number to one counted by the larger number would be nice but a better proof would be an expression for the difference as a sum of obviously positive terms. For $i = 1$, we have

$$\binom{3a+1}{a+1} - \binom{3a+2}{a} = \binom{3a+1}{a-1},$$

and the right-hand side is clearly positive. It seems for general $i = 1, 2, \dots$,

$$\binom{3a+i}{a+i} - \binom{3a+2i}{a} = \sum_{k=1}^i c_k(i) \binom{3a+i}{a-k}$$

for integers $c_k(i)$ and, furthermore, the $c_k(i)$ are all positive. Here is a table for $c_k(i)$ when $1 \leq k \leq i \leq 8$:

1						
3	1					
6	6	1				
10	19	9	1			
15	45	39	12	1		
21	90	120	66	15	1	
28	161	301	250	100	18	1
36	266	658	755	450	141	21

but appeared hard to get a handle on them. The evolution of our next progress begins with the discovery of

$$c_k(i) = \binom{i+k-1}{2k} + 2 \binom{i+k-1}{2k-1} - \binom{i}{k}.$$

Let's contract these coefficients as $c_k(i) = \frac{i+3k}{i+k} \binom{i+k}{2k} - \binom{i}{k}$, for $i, k \geq 1$. Notice $c_0(i) = 0$. We need some preliminary results.

Lemma 1. *We have*

$$\binom{3a+2i}{a} = \sum_{k \geq 0} \binom{i}{k} \binom{3a+i}{a-k}.$$

Proof. This follows from the Vandermonde-Chu identity (Theorem 1 for $q = 1$)

$$\binom{X+Y}{Z} = \sum_{k \geq 0} \binom{X}{k} \binom{Y}{Z-k}$$

applied to $\binom{3a+2i}{a} = \binom{i+3a+i}{a}$ with $X = i, Y = 3a + i$ and $Z = a$. $\square$

Lemma 2. *We have*

$$\binom{3a+i}{a+i} = \sum_{k \geq 0} \frac{i+3k}{i+k} \binom{i+k}{2k} \binom{3a+i}{a-k} = \sum_{k \geq 0} \left[\binom{i+k}{2k} + \binom{i+k-1}{2k-1} \right] \binom{3a+i}{a-k}.$$

Proof. We implement Zeilberger's algorithm (from the Wilf-Zeilberger theory). Define

$$F(i, k) = \frac{i+3k}{i+k} \cdot \frac{\binom{i+k}{2k} \binom{3a+i}{a-k}}{\binom{3a+i}{a+i}} \quad \text{and} \quad G(i, k) = -\frac{\binom{i+k-1}{2k-2} \binom{3a+i}{a-k}}{\binom{3a+i}{a+i}}.$$

Check that $F(i+1, k) - F(i, k) = G(i, k+1) - G(i, k)$ and sum both sides over all integer values k . Then, notice the right-hand side vanishes and hence we obtain a sum $\sum_k F(i, k)$ that is *constant* in the variable i . Determine this constant by substituting, say $i = 1$,

$$\sum_{k=0}^1 F(1, k) = \frac{\binom{3a+1}{a}}{\binom{3a+1}{a+1}} + \frac{2\binom{3a+1}{a-1}}{\binom{3a+1}{a+1}} = \frac{a+1}{2a+1} + \frac{a}{2a+1} = 1.$$

Therefore, $\sum_k F(i, k) = 1$, identically, for all $i \geq 1$. The proof follows. $\square$

We now state the main result of this section.

Theorem 3. *We have*

$$\binom{3a+i}{a+i} - \binom{3a+2i}{a} = \sum_{k \geq 1} \left\{ \frac{i+3k}{i+k} \binom{i+k}{2k} - \binom{i}{k} \right\} \binom{3a+i}{a-k}.$$

Proof. Immediate from Lemma 1 and Lemma 2. $\square$

Lemma 3. *For $k \geq 1$, the coefficients $c_k(i)$ are non-negative.*

Proof. We may look at it in two different ways:

(1) $c_k(i) = \frac{2k}{i+k} \binom{i+k}{2k} + \binom{i+k}{2k} - \binom{i}{k} = \frac{2k}{i+k} \binom{i+k}{2k} + \binom{i+k}{i-k} - \binom{i}{i-k}$. Obviously $\binom{i+k}{i-k} \geq \binom{i}{i-k}$, therefore $c_k(i) \geq 0$.

(2) $c_k(i) = \binom{i+k}{2k} + \binom{i+k-1}{2k-1} - \binom{i}{k} = \binom{i+k}{2k} + \sum_{r=0}^{k-2} \binom{i+r}{k+1+r}$ shows clearly that $c_k(i) \geq 0$. The identity $\binom{i+k-1}{2k-1} = \binom{i}{k} + \sum_{r=1}^{k-1} \binom{i+r-1}{k+r}$ results from a cascading effect of the familiar binomial recurrence $\binom{u}{v} + \binom{u}{v-1} = \binom{u+1}{v}$. $\square$

3. q -ANALOGUES WHEN $\beta = 2$

In the present section, we aim to generalize our proofs given in the preceding section by lifting the argument from the ordinary binomials to Gaussian polynomials.

Lemma 4. *We have*

$$\binom{3a+2i}{a}_q = \sum_{k \geq 0} q^{(a-k)(i-k)} \binom{i}{k}_q \binom{3a+i}{a-k}_q.$$

Proof. This follows from the *Vandermonde-Chu* identity (Theorem 1)

$$\binom{X+Y}{Z}_q = \sum_{k \geq 0} q^{(Z-k)(X-k)} \binom{X}{k}_q \binom{Y}{Z-k}_q$$

on $\binom{3a+2i}{a}_q = \binom{i+3a+i}{a}_q$ with $X = i, Y = 3a + i$ and $Z = a$. □

Lemma 5. *We have*

$$\binom{3a+i}{a+i}_q = \sum_{k \geq 0} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q \right] \binom{3a+i}{a-k}_q.$$

Proof. Let's rewrite $\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q = \left[1 + \frac{q^{a+i}(1-q^{2k})}{1-q^{i+k}} \right] \binom{i+k}{2k}_q$ and define the functions

$$F(i, k) = q^{(a-k)(i-k)} \left[1 + q^{a+i} \cdot \frac{1-q^{2k}}{1-q^{i+k}} \right] \frac{\binom{i+k}{2k}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}, \quad \text{and}$$

$$G(i, k) = -q^{(a-k+1)(i-k+1)} \cdot \frac{\binom{i+k-1}{2k-2}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}.$$

Divide both sides of the intended identity by $\binom{3a+i}{a+i}_q$. Our goal is to prove $\sum_k F(i, k) = 1$ by adopting the Wilf-Zeilberger technique. To this end, calculate the following two ratios

$$A(i, k) := \frac{F(i+1, k)}{F(i, k)} - 1 \quad \text{and} \quad B(i, k) := \frac{G(i, k+1)}{F(i, k)} - \frac{G(i, k)}{F(i, k)}$$

resulting in

$$A(i, k) = \frac{q^{a-k}(1-q^{i+k})(1-q^{a+i+1})(1-q^{i+k+1}+q^{a+i+1}-q^{a+i+2k+1})}{(1-q^{i-k+1})(1-q^{2a+i+k+1})(1-q^{i+k}+q^{a+i}-q^{a+i+2k})} - 1 \quad \text{and}$$

$$B(i, k) = \left[-\frac{1-q^{a-k}}{1-q^{2a+i+k+1}} + \frac{q^{a+i-2k+1}(1-q^{2k})(1-q^{2k-1})}{(1-q^{i+k})(1-q^{i-k+1})} \right] \cdot \frac{1-q^{i+k}}{1-q^{i+k}+q^{a+i}-q^{a+i+2k}}.$$

Verify routinely $A(i, k) = B(i, k)$. Thus $F(i+1, k) - F(i, k) = G(i, k+1) - G(i, k)$. Now, sum both sides over all integer values k . Then, notice that the right-hand side vanishes

and hence we obtain a sum $\sum_k F(i, k)$ that is *constant* in the variable i . Determine this constant by substituting, say $i = 1$ and proceed with some simplifications leading to

$$\sum_{k=0}^1 F(1, k) = q^a \cdot \frac{1 - q^{a+1}}{1 - q^{2a+1}} + \frac{1 - q^a}{1 - q^{2a+1}} = 1.$$

Therefore, $\sum_k F(i, k) = 1$, identically, for all $i \geq 1$. The assertion follows. $\square$

Lemma 6. *We have the identity*

$$\binom{i+k}{2k}_q = \binom{i}{k}_q + \sum_{r=1}^k q^{k+r} \binom{i+r-1}{k+r}_q.$$

Proof. Use the recursive relations $\binom{a}{b}_q = \binom{a-1}{b}_q + q^{a-b} \binom{a-1}{b-1}_q = q^b \binom{a-1}{b}_q + \binom{a-1}{b-1}_q$. $\square$

Lemma 7. *We have the inequality $\binom{i+k}{i-k}_q \geq \binom{i}{i-k}_q$.*

Proof. We use the interpretation of the Gaussian polynomials as the *inversion number* generating function for all bit strings of length n with k zeroes and $n - k$ ones, that is

$$\binom{n}{k}_q = \sum_{w \in 0^k 1^{n-k}} q^{\text{inv}(w)}.$$

Let $w' \in 0^{i-k} 1^k \sqcup 1^k$ denote a bit where the last k digits are all ones. In this sense, we get

$$\begin{aligned} \binom{i+k}{i-k}_q &= \sum_{w \in 0^{i-k} 1^{2k}} q^{\text{inv}(w)} = \sum_{w' \in 0^{i-k} 1^k \sqcup 1^k} q^{\text{inv}(w')} + \sum_{w' \notin 0^{i-k} 1^k \sqcup 1^k} q^{\text{inv}(w')} \\ &= \sum_{w \in 0^{i-k} 1^k} q^{\text{inv}(w)} + \sum_{w' \notin 0^{i-k} 1^k \sqcup 1^k} q^{\text{inv}(w')} \\ &= \binom{i}{i-k}_q + \sum_{w' \notin 0^{i-k} 1^k \sqcup 1^k} q^{\text{inv}(w')} \end{aligned}$$

where we note that $\text{inv}(w') = \text{inv}(w)$ if the word $w' \in 0^{i-k} 1^k \sqcup 1^k$ is associated with $w \in 0^{i-k} 1^k$ found by dropping the last k ones. The assertion is now immediate. $\square$

We prove the main result of this section and our paper, the β -Conjecture for $\beta = 2$.

Theorem 4. *The polynomial $P(q) := \binom{3a+i}{a+i}_q - \binom{3a+2i}{a}_q$ has non-negative coefficients.*

Proof. From Lemma 4 and Lemma 5, we infer

$$P(q) = \sum_{k \geq 1} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q - \binom{i}{k}_q \right] \binom{3a+i}{a-k}_q.$$

It suffices to verify positivity of the terms inside the inner parenthesis on the right-hand side. We may pair up these terms and compliment the lower index to the effect that

$$\binom{i+k}{2k}_q - \binom{i}{k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q = \binom{i+k}{i-k}_q - \binom{i}{i-k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q.$$

To reach the conclusion, simply apply Lemma 6 or Lemma 7. $\square$

4. FINAL REMARKS

In the present section, we close our discussion with one conjecture as a codicil of certain calculations we encountered while digging up ways to prove the β -Conjecture.

Conjecture 4. *For each $0 \leq k \leq a < b$, we have*

$$\binom{a}{k}_q \binom{a+b}{b-k}_q \geq \binom{b}{k}_q \binom{a+b}{a-k}_q \quad \text{or} \quad \binom{a}{k}_q \binom{b}{k}_q \binom{b+a}{b}_q \left[\frac{1}{\binom{a+k}{k}_q} - \frac{1}{\binom{b+k}{k}_q} \right] \geq 0.$$

The next elementary result might be helpful if one decides to engage this conjecture.

Lemma 8. *For $0 \leq k \leq a < b$, we have*

$$\frac{1}{\binom{a+k}{k}_q} - \frac{1}{\binom{b+k}{k}_q} = \sum_{i=1}^k q^{a+i} \frac{1 - q^{b-a}}{1 - q^{b+i}} \prod_{j=i}^k \frac{1 - q^j}{1 - q^{a+j}} \prod_{j=1}^{i-1} \frac{1 - q^j}{1 - q^{b+j}}.$$

Proof. This results from partial fractions. $\square$

Example 1.

$$\begin{aligned} \frac{1}{\binom{a+1}{1}_q} - \frac{1}{\binom{b+1}{1}_q} &= \frac{q^{a+1}(1-q)(1-q^{b-a})}{(1-q^{a+1})(1-q^{b+1})} \\ \frac{1}{\binom{a+2}{2}_q} - \frac{1}{\binom{b+2}{2}_q} &= \frac{q^{a+1}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{b+1})} + \frac{q^{a+2}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+2})(1-q^{b+1})(1-q^{b+2})}. \end{aligned}$$

Example 2.

$$\begin{aligned} \frac{1}{\binom{a+3}{3}_q} - \frac{1}{\binom{b+3}{3}_q} &= \frac{q^{a+1}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})} \\ &\quad + \frac{q^{a+2}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})} \\ &\quad + \frac{q^{a+3}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})(1-q^{b+3})}. \end{aligned}$$

Remark 5. As a side note, we recall that G. E. Andrews [2] expresses $\binom{n}{k}_q - \binom{n}{k-1}_q$ as the generating function for partitions with particular *Frobenius symbols*, while L. M. Butler [5] does this with the help of the *Kostka-Foulkes polynomials* to show non-negativity of the coefficients. We shall provide an alternative algebraic approach.

Lemma 9. For $0 \leq 2k \leq n$, we have $\binom{n}{k}_q - \binom{n}{k-1}_q \geq 0$.

Proof. Let $n = \alpha k + d$ where $0 \leq d < k$. Rewrite

$$\binom{n}{k}_q - \binom{n}{k-1}_q = q^k \binom{n}{k-1}_q \frac{1 - q^{(\alpha-2)k}}{1 - q^k} + q^{(\alpha-1)k} \binom{n}{k-1}_q \frac{1 - q^{d+1}}{1 - q^k}.$$

Observe $\frac{1 - q^{(\alpha-2)k}}{1 - q^k}$ is already a polynomial with non-negative coefficients. Furthermore, since $U(q) := \binom{n}{k-1}_q$ is unimodal [1], [6], [7], the coefficient of q^j in $U(q) \cdot (1 - q^{d+1})$ is non-negative as long as $2j \leq \deg(U)$. The same is true for $U(q) \frac{1 - q^{d+1}}{1 - q^k}$ as a formal power series. Since the polynomial $U(q) \frac{1 - q^{d+1}}{1 - q^k}$ is *symmetric*, having degree no greater than $U(q)$, all remaining coefficients of $U(q) \frac{1 - q^{d+1}}{1 - q^k}$ are non-negative. $\square$

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