Jordan and Smith forms of Pascal-related matrices

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1. Definitions Let $P_n = \left(\binom{i-1}{j-1}\right)_{1\le i,j\le n}$, $S_n = \left(\left\{{i-1 \atop j-1}\right\}\right)_{1\le i,j\le n}$, $C_n = \left(\left[{i-1 \atop j-1}\right]\right)_{1\le i,j\le n}$ denote the n-by-n lower triangular matrices of binomial coefficients, Stirling partition numbers and Stirling cycle numbers respectively. For example,

$$P_5 = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 2 & 1 && \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & 1 & 1 && \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & 1 & 1 && \\ 0 & 2 & 3 & 1 & \\ 0 & 6 & 11 & 6 & 1 \end{pmatrix}.$$

We use $i^{\underline{j}}$ and $i^{\overline{j}}$ for the falling and rising factorials respectively and we adopt the convention a binomial coefficient is zero if either of its parameters is negative. Let $F_{n,r}$ denote the n-by-n matrix $\left((i-1)^{\underline{i-j}}\binom{i-j-1}{r-1}\right)_{1\le i,j\le n}$. Note that $F_{n,r}$ is a block matrix $\left({0 \atop G_{n,r}}\ {0 \atop 0}\right)$ with $G_{n,r}$ lower triangular of size $n-r$. Let $H_{n,r}$ denote the $(n-r)$-by-$(n-r)$ lower triangular banded matrix $\left((-1)^{(i-j)}\binom{i-1}{j-1}r^{\underline{i-j}}\right)_{1\le i,j\le n-r}$. Note the (i,j) entry of $H_{n,r}$ is 0 if $i-j>r$ and can also be expressed as $\binom{i-1}{j-1}(-r)^{\overline{i-j}}$. Let $D_{n,r}$ denote the diagonal matrix with the same diagonal, $(i^{\overline{r}})_{1\le i\le n-r}$, as $G_{n,r}$. For example,

$$G_{6,2} = \begin{pmatrix} 2 &&& \\ 12 & 6 && \\ 72 & 48 & 12 & \\ 480 & 360 & 120 & 20 \end{pmatrix}, \quad H_{6,2} = \begin{pmatrix} 1 &&& \\ -2 & 1 && \\ 2 & -4 & 1 & \\ 0 & 6 & -6 & 1 \end{pmatrix}, \quad D_{6,2} = \begin{pmatrix} 2 &&& \\ & 6 && \\ && 12 & \\ &&& 20 \end{pmatrix}.$$

2. Matrix Identities With the preceding definitions, we have the following four identities:

$$S_n^{-1}\,P_n\,S_n = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 2 & 1 & & \\ & & \ddots & \ddots & \\ & & & n-1 & n \end{pmatrix} \tag{1}$$

$$\left(\begin{bmatrix} i \\ j \end{bmatrix}\right)_{1\le i,j\le n} \left(\binom{i}{j}\right)_{1\le i,j\le n} \left(\begin{bmatrix} i \\ j \end{bmatrix}\right)^{-1}_{1\le i,j\le n} = \mathrm{diag}\Big((1,2,\ldots,n)\Big) \tag{2}$$

$$C_n\,(P_n - I_n)^r\,C_n^{-1} = F_{n,r} \tag{3}$$

$$G_{n,r}\,H_{n,r} = D_{n,r} \tag{4}$$

3. Canonical Forms Identity (1) puts P_n in bidiagonal form, which is good enough to obtain the Jordan form: say a matrix is near-Jordan if it has the form

$$J_n(\lambda; c_1,\ldots,c_{n-1}) = \begin{pmatrix} \lambda & & & & \\ c_1 & \lambda & & & \\ & c_2 & \lambda & & \\ & & \ddots & \ddots & \\ & & & c_{n-1} & \lambda \end{pmatrix}$$

with all the c_i's nonzero. (A Jordan matrix is one with each $c_i = 1$.) Now $J_n(\lambda; c_1,\ldots,c_{n-1})$ is similar to a Jordan matrix via

$$D\,J_n(\lambda; c_1,\ldots,c_{n-1})\,D^{-1} = J_n(\lambda; 1,\ldots,1)$$

where $D = \mathrm{diag}\Big((1, c_1, c_1c_2, \ldots, c_1c_2\cdots c_{n-1})\Big)$. Hence the Jordan form of P_n consists of a single block with $\lambda = 1$. We can deduce the Jordan form of P_n mod p: reducing (1) mod p, the right side becomes block diagonal with as many p-by-p near-Jordan blocks as will fit in an n-by-n matrix and (possibly) one of smaller size. Hence we have

Theorem 1 *The Jordan form of P_n mod p has precisely $\lceil \frac{n}{p} \rceil$ blocks (all corresponding to the eigenvalue 1) and $\lfloor \frac{n}{p} \rfloor$ of them are of size p.*

In particular, for $n \geq p$ the minimal polynomial of P_n mod p is $(x-1)^p$ [2].

Identity (2) shows

Theorem 2 *The matrix* $\left(\begin{bmatrix} i \\ j \end{bmatrix} \right)^{-1}_{1 \leq i,j \leq n} = \left((-1)^{i-j} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} \right)_{1 \leq i,j \leq n}$ *is a matrix of eigenvectors for* $\left(\binom{i}{j} \right)_{1 \leq i,j \leq n}$.

Recall that two integer matrices A, B are *equivalent* if there exist unimodular matrices U, V such that $UAV = B$. From (4), we have $\begin{pmatrix} 0 & I_{n-r} \\ I_r & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ G_{n,r} & 0 \end{pmatrix} \begin{pmatrix} H_{n,r} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{n,r} & 0 \\ 0 & 0 \end{pmatrix}$ where the middle matrix in the product is $F_{n,r}$. Combined with (3), we have

Theorem 3 $(P_n - I_n)^r$ *is equivalent to the diagonal matrix diag*$(1^{\overline{r}}, 2^{\overline{r}}, \ldots, (n-r)^{\overline{r}}, 0, \ldots, 0)$.

This is good enough [1, Theorem II.13, p. 30] to obtain the elementary divisors and hence the Smith normal form of $(P_n - I_n)^r$.

4. Proofs After left multiplying by S_n, equating the $(n+1, m+1)$ entries and using the defining recurrence $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ m \end{matrix} \right\} + (m+1) \left\{ \begin{matrix} n \\ m+1 \end{matrix} \right\}$, identity (1) reduces to $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\}$ [3, 6.15, p. 265]. Similarly, identity (2) is equivalent to a simple variant of $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m}$ [3, 6.16, p. 265]

Equating the (n, m) entries, (3) is equivalent to

$$\sum_{k=m+r}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} = \sum_{k=m}^{n-r} n^{\underline{n-k}} \binom{n-k-1}{r-1} \begin{bmatrix} k \\ m \end{bmatrix}.$$

Both sides count the following set S of combinatorial objects: an object of S is a partition of $[n]$ into cycles, m of which are colored black (say) and the rest are colored from $\{1, 2, \ldots, r\}$ so that all r colors appear. The left side counts S by number of cycles: $\begin{bmatrix} n \\ k \end{bmatrix}$—partition $[n]$ into k cycles; $\binom{k}{m}$—choose the black cycles; $r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\}$—assign colors to the other $k-m$ cycles. The right side counts S by total number of elements in the black cycles: $n^{\underline{n-k}}$—form a list of $n-k$ elements from $[n]$; $\binom{n-k-1}{r-1}$—choose $r-1$ of the $n-k-1$ "dividers" separating elements of the list, thereby obtaining a list of r nonempty lists each of which is canonically converted to cycles (for example, canonical might mean that cycles start at the left to right minima of a list), all cycles from the ith sublist getting color i; $\left[{k \atop m}\right]$—choose m black cycles on the remaining k elements.

Identity (4) is a consequence of the binomial theorem for rising factorials which implies that the inverse of $H_{n,r} = \left(\binom{i-1}{j-1}(-r)^{\overline{i-j}}\right)_{1\le i,j\le n-r}$ is $\left(\binom{i-1}{j-1}r^{\overline{i-j}}\right)_{1\le i,j\le n-r}$.

5. Further Remarks The (i,j) entry of $(P_n - I_n)^r$ has a closed form, $r!\left\{{i-j \atop r}\right\}\binom{i-1}{j-1}$, for which there is a simple combinatorial explanation. For a sequence $\mathbf{c} = (c_i)_{i\ge 0}$, set $P_n(\mathbf{c}) = \left(c_{i-j}\binom{i-1}{j-1}\right)_{1\le i,j\le n}$. Then $P_n(\mathbf{c})P_n(\mathbf{d}) = P_n(\mathbf{c}*\mathbf{d})$ where $\mathbf{c}*\mathbf{d} = \left(\sum_{i=0}^{n}\binom{n}{i}c_i d_{n-i}\right)_{n\ge 0}$. If $\mathbf{c}$ counts labeled structures of some kind (c_n is the number of structures on $[n]$), then the r-fold convolution $\mathbf{c}*\ldots*\mathbf{c}$ counts r-sequences of structures whose labels partition $[n]$. In particular, for (the species of) nonempty sets $\mathbf{c} = (0,1,1,1,\ldots)$ and $\mathbf{c}*\ldots*\mathbf{c}$ counts ordered partitions of $[n]$ into r (unordered nonempty) blocks; the number of such ordered partitions is $r!\left\{{n \atop r}\right\}$. Hence, with $\mathbf{c} := (0,1,1,1,\ldots)$ and $\mathbf{d} := \left(r!\left\{{i \atop r}\right\}\right)_{i\ge 0}$, $(P_n - I_n)^r = P_n(\mathbf{c})^r = P_n(\mathbf{c}*\ldots*\mathbf{c}) = P_n(\mathbf{d}) = \left(r!\left\{{i-j \atop r}\right\}\binom{i-1}{j-1}\right)_{1\le i,j\le n}$, as asserted.

Say an integer matrix is equivalent to its diagonal if it is equivalent to the matrix obtained by zeroing out all offdiagonal entries. Suppose a sequence $\mathbf{c}$ (such as $\left(\left\{{i \atop r}\right\}\right)_{i\ge 0}$) begins with r 0's so that $P_n(\mathbf{c})$ has the block form $\begin{pmatrix} 0 & 0 \\ Q_n(\mathbf{c}) & 0 \end{pmatrix}$ with $Q_n(\mathbf{c})$ square of size $n-r$. It follows from identities (3) and (4) above that the following statement is true. For $\mathbf{c} = \left(\left\{{i \atop r}\right\}\right)_{i\ge 0}$, $Q_n(\mathbf{c})$ is equivalent to its diagonal. It seems the same statement is true for $\mathbf{c} = \left(\left[{i \atop r}\right]\right)_{i\ge 0}$ and it would be interesting to have a proof.

References

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[3] Graham, Knuth, Patashnik, *Concrete Mathematics* (2nd edition), Addison-Wesley, 1994.