Two Bijections for Dyck Path Parameters

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The Motzkin number M_n (A001006) counts Motzkin n-paths: lattice paths of upsteps $U = (1,1)$, flatsteps $F = (1,0)$ and downsteps $D = (1,-1)$ such that (i) the path contains n steps, (ii) the number of Us = number of Ds, and (iii) the path never dips below the horizontal line joining its initial and terminal points (ground level) [1]. A Dyck n-path is a Motzkin $(2n)$-path with no flatsteps, counted by the Catalan number C_n. A UUU-free Dyck path is one that contains no consecutive UUU and so on. A descent in a path is a maximal sequence of contiguous downsteps. A short descent is one consisting of a single downstep. A terminal descent is one that ends the path. Thus a Dyck path is UDU-free iff it contains no short nonterminal descents. Each upstep in a Dyck (or Motzkin) path has a matching downstep: the first one encountered directly East of the upstep.

The first result is easy, and also appears in [2].

Theorem 1. *The number of UUU-free Dyck n-paths is M_n.*

Proof. Given a UUU-free Dyck n-path, change each UUD to U, then change each remaining UD to F. This is a bijection to Motzkin n-paths. For example with $n = 5$: $U\,U\,D\,U\,D\,U\,U\,D\,D\,D -> U\,F\,U\,D\,D$. The inverse is obvious. □

The next bijection gives the distributions both of the parameter "number of UDUs" (A091869) [3] and the parameter "number of DDUs" (A091894) on Dyck paths.

Theorem 2.

(*i*) *The number of Dyck n-paths containing exactly k UDUs is* $\binom{n-1}{k} M_{n-1-k}$ (*Donaghey distribution*).

(*ii*) *The number of Dyck n-paths containing exactly k DDUs is* $\binom{n-1}{2k} 2^{n-1-2k} C_k$ (*Touchard distribution*).

Proof. A bicolored Motzkin n-path is a Motzkin path of length n in which each flatstep is given one of two colors, say green and black, denoted by wavy and flat overlines respectively. There is an obvious bijection to Dyck $(n+1)$-paths: replace each U by UU, D by DD, $\overline{F}$ by UD, $\widetilde{F}$ by UD and then prepend a U and append a D.

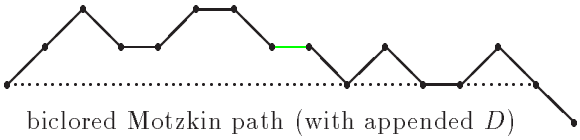
Here is a less obvious bijection that takes $\#\widetilde{F}$s to $\#UDU$s and $\#D$s to $\#DDU$s. Since it is easy to count bicolored Motzkin paths either by $\#\widetilde{F}$s or by $\#D$s, we get the stated distributions.

Given a bicolored Motzkin n-path, first append a downstep as a convenience. Now every flatstep F has an associated downstep: the first D whose initial point is on the ray obtained by extending F eastward. Then

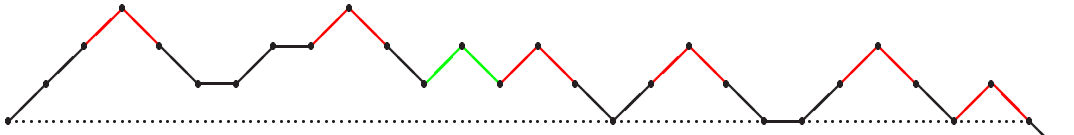
1. Leave Us intact
2. Replace D by UDD
3. Replace $\widetilde{F}$ by UD
4. Replace $\overline{F}$ by U and insert a D immediately before its associated downstep (thus the new U and inserted D will be a matching UD pair and the result is the same no matter in what order the $\overline{F}$s are processed).

Finally, delete the appended D (it remains undisturbed). An example with $n = 14$ is

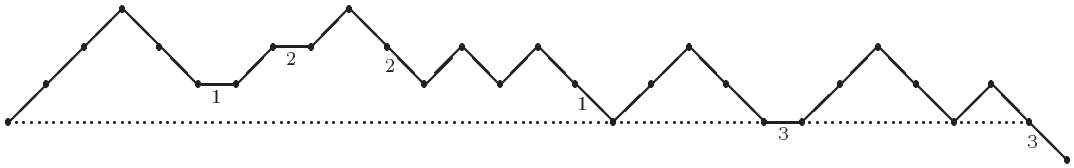
illustrated.



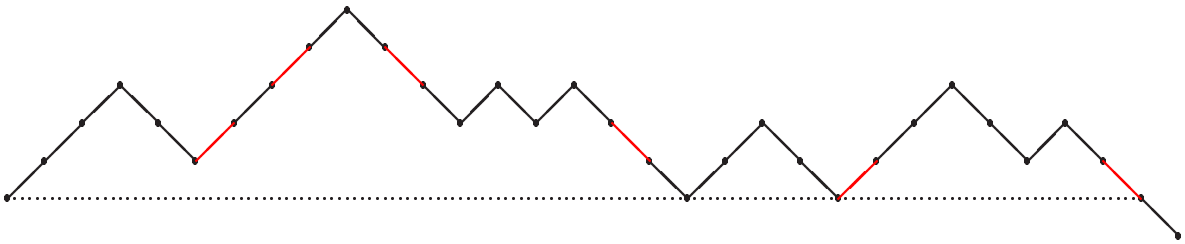
biclored Motzkin path (with appended D)



UD (in red) inserted before each D
green Fs become UDs



remaining Fs labeled along with associated Ds



change Fs to Us and insert matching Ds
(changed and inserted steps in red)

The map is invertible: $\widetilde{F}$s are recaptured as UDs followed by a U, Ds are recaptured as UDDs, $\overline{F}$s are recaptured as Us whose matching D is in the *interior* of a descent of length ≥ 3 and the original Us are recaptured as Us such that both the U itself and its matching D are followed by a U. □

This map restricted to Motzkin paths (no green Fs) is a bijection from Motzkin n-paths to UDU-free Dyck $(n+1)$-paths. Further, note that a Motzkin path contains no

flatsteps at ground level iff the corresponding UDU-free Dyck path ends with UD. So, by deleting this UD, Motzkin n-paths with no flatsteps at ground level (A005043) correspond to Dyck n-paths with no short descents.

References

[1] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999.

[2] Sergi Elizalde and Toufik Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, preprint.

[3] Emeric Deutsch and Louis W.Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, Discrete Math. **256** (2002), 655-670.