

# An Involutory Matrix of Eigenvectors

David Callan,<sup>1</sup>

Department of Statistics, University of Wisconsin-Madison,  
1210 W. Dayton Street, Madison, WI 53706-1693, U.S.A.

Helmut Prodinger,

The John Knopfmacher Centre for Applicable Analysis and Number Theory,  
School of Mathematics, University of the Witwatersrand,  
P. O. Wits, 2050 Johannesburg, South Africa.

A matrix with a full set of linearly independent eigenvectors is diagonalizable: if the  $n$  by  $n$  matrix  $A$  has eigenvalues  $\lambda_j$  with corresponding eigenvectors  $u_j$  ( $1 \leq j \leq n$ ), if  $U = (u_1|u_2|\dots|u_n)$  and  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ , then  $U$  is a diagonalizing matrix for  $A$ :  $U^{-1}AU = D$ . Taking transposes shows that  $(U^{-1})^t$  is a diagonalizing matrix for  $A^t$ . Hence  $U^t$  itself is a diagonalizing matrix for  $A^t$  if  $U^2$  is the identity matrix, or more generally, due to the scalability of eigenvectors, if  $U^2$  is a scalar matrix.

The purpose of this note is to point out that the right-justified Pascal-triangle matrix  $R = \left( \binom{i-1}{n-j} \right)_{1 \leq i, j \leq n}$  is an example of this phenomenon. Let  $a$  denote the golden ratio  $(1 + \sqrt{5})/2$ . The eigenvalues of  $R^t$  (which of course are the same as the eigenvalues of  $R$ ) were found in [1]:  $\lambda_i = (-1)^{n-i} a^{2i-n-1}$ ,  $1 \leq i \leq n$ . The corresponding eigenvectors  $u_i$  of  $R^t$  were also found in [1] (here suitably scaled for our purposes):  $u_i = (u_{ij})_{1 \leq j \leq n}$  where

$$u_{ij} = (-a)^{n-j} \sum_{k=1}^j (-1)^{i-k} \binom{i-1}{k-1} \binom{n-i}{j-k} a^{2k-i-1}.$$

Let  $U = (u_{ij})_{1 \leq i, j \leq n}$ .

For example, when  $n = 5$ ,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} a^4 & -4a^3 & 6a^2 & -4a & 1 \\ -a^3 & 3a^2 - a^4 & -3a + 3a^3 & 1 - 3a^2 & a \\ a^2 & -2a + 2a^3 & 1 - 4a^2 + a^4 & 2a - 2a^3 & a^2 \\ -a & 1 - 3a^2 & 3a - 3a^3 & 3a^2 - a^4 & a^3 \\ 1 & 4a & 6a^2 & 4a^3 & a^4 \end{pmatrix}.$$

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<sup>1</sup>corresponding author

Since the rows of  $U$  are eigenvectors of  $R^t$ ,  $U^t$  is a diagonalizing matrix for  $R^t$ . By the first paragraph applied to  $A = R^t$ ,  $U$  will be a diagonalizing matrix for  $R$  if we can show that  $(U^t)^2$  (equivalently  $U^2$ ) is a scalar matrix. We now proceed to show that  $U^2 = (1+a^2)^{n-1}I_n$  and in fact this holds for arbitrary  $a$ . We use the notation  $[x^k]p(x)$  to denote the coefficient of  $x^k$  in the polynomial  $p(x)$ . Consider the generating function  $U_i(z) = z(z-a)^{n-i}(az+1)^{i-1}$ . Using the binomial theorem to expand  $U_i(z)$ , it is immediate that

$$U_i(z) = \sum_{j=1}^n u_{ij} z^j.$$

Now the  $(i, k)$  entry of  $U^2$  is

$$\begin{aligned} (U^2)_{ik} &= \sum_{j=1}^n [x^{k-1}](x-a)^{n-j}(ax+1)^{j-1} \cdot [z^{j-1}](z-a)^{n-i}(az+1)^{i-1} \\ &= [x^{k-1}] \sum_{j=1}^n [z^{j-1}](x-a)^{n-j}(ax+1)^{j-1}(z-a)^{n-i}(az+1)^{i-1} \\ &= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^n [z^{j-1}] \left( \frac{ax+1}{x-a} \right)^{j-1} (z-a)^{n-i}(az+1)^{i-1} \\ &= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^n [z^{j-1}] \left( z \frac{ax+1}{x-a} - a \right)^{n-i} \left( a z \frac{ax+1}{x-a} + 1 \right)^{i-1} \\ &= [x^{k-1}](x-a)^{n-1} \left( \frac{ax+1}{x-a} - a \right)^{n-i} \left( a \frac{ax+1}{x-a} + 1 \right)^{i-1} \\ &= [x^{k-1}](ax+1-ax+a^2)^{n-i} (a^2x+a+x-a)^{i-1} \\ &= [x^{k-1}](1+a^2)^{n-i} x^{i-1} (1+a^2)^{i-1} \\ &= [x^{k-i}](1+a^2)^{n-1} \\ &= (1+a^2)^{n-1} \delta_{ki}, \end{aligned}$$

as desired.

## References

- [1] R. S. Melham and Curtis Cooper. "The Eigenvectors of a Certain Matrix of binomial Coefficients." *The Fibonacci Quarterly* **38.2** (2000):123-26.

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