An Involutory Matrix of Eigenvectors

David Callan,[1]
Department of Statistics, University of Wisconsin-Madison,
1210 W. Dayton Street, Madison, WI 53706-1693, U.S.A.

Helmut Prodinger,
The John Knopfmacher Centre for Applicable Analysis and Number Theory,
School of Mathematics, University of the Witwatersrand,
P. O. Wits, 2050 Johannesburg, South Africa.

A matrix with a full set of linearly independent eigenvectors is diagonalizable: if the n by n matrix A has eigenvalues λ_j with corresponding eigenvectors u_j $(1 \le j \le n)$, if $U = (u_1|u_2|\ldots|u_n)$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then U is a diagonalizing matrix for A: $U^{-1}AU = D$. Taking transposes shows that $(U^{-1})^t$ is a diagonalizing matrix for A^t. Hence U^t itself is a diagonalizing matrix for A^t if U^2 is the identity matrix, or more generally, due to the scalability of eigenvectors, if U^2 is a scalar matrix.

The purpose of this note is to point out that the right-justified Pascal-triangle matrix $R = \left(\binom{i-1}{n-j}\right)_{1\le i,j\le n}$ is an example of this phenomenon. Let a denote the golden ratio $(1+\sqrt{5})/2$. The eigenvalues of R^t (which of course are the same as the eigenvalues of R) were found in [1]: $\lambda_i = (-1)^{n-i}a^{2i-n-1}$, $1 \le i \le n$. The corresponding eigenvectors u_i of R^t were also found in [1] (here suitably scaled for our purposes): $u_i = (u_{ij})_{1\le j\le n}$ where

$$u_{ij} = (-a)^{n-j}\sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k}a^{2k-i-1}.$$

Let $U = (u_{ij})_{1\le i,j\le n}$.

For example, when $n = 5$,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} a^4 & -4\,a^3 & 6\,a^2 & -4\,a & 1 \\ -a^3 & 3\,a^2 - a^4 & -3\,a + 3\,a^3 & 1 - 3\,a^2 & a \\ a^2 & -2\,a + 2\,a^3 & 1 - 4\,a^2 + a^4 & 2\,a - 2\,a^3 & a^2 \\ -a & 1 - 3\,a^2 & 3\,a - 3\,a^3 & 3\,a^2 - a^4 & a^3 \\ 1 & 4\,a & 6\,a^2 & 4\,a^3 & a^4 \end{pmatrix}.$$

[1]corresponding author

Since the rows of U are eigenvectors of R^t, U^t is a diagonalizing matrix for R^t. By the first paragraph applied to $A = R^t$, U will be a diagonalizing matrix for R if we can show that $(U^t)^2$ (equivalently U^2) is a scalar matrix. We now proceed to show that $U^2 = (1+a^2)^{n-1} I_n$ and in fact this holds for arbitrary a. We use the notation $[x^k]p(x)$ to denote the coefficient of x^k in the polynomial $p(x)$. Consider the generating function $U_i(z) = z(z-a)^{n-i}(az+1)^{i-1}$. Using the binomial therem to expand $U_i(z)$, it is immediate that

$$U_i(z) = \sum_{j=1}^{n} u_{ij} z^j.$$

Now the (i,k) entry of U^2 is

$$\begin{aligned}
(U^2)_{ik} &= \sum_{j=1}^{n} [x^{k-1}](x-a)^{n-j}(ax+1)^{j-1} \cdot [z^{j-1}](z-a)^{n-i}(az+1)^{i-1} \\
&= [x^{k-1}] \sum_{j=1}^{n} [z^{j-1}](x-a)^{n-j}(ax+1)^{j-1}(z-a)^{n-i}(az+1)^{i-1} \\
&= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^{n} [z^{j-1}] \left(\frac{ax+1}{x-a}\right)^{j-1} (z-a)^{n-i}(az+1)^{i-1} \\
&= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^{n} [z^{j-1}] \Big(z\frac{ax+1}{x-a} - a\Big)^{n-i} \Big(az\frac{ax+1}{x-a} + 1\Big)^{i-1} \\
&= [x^{k-1}](x-a)^{n-1} \Big(\frac{ax+1}{x-a} - a\Big)^{n-i} \Big(a\frac{ax+1}{x-a} + 1\Big)^{i-1} \\
&= [x^{k-1}](ax+1-ax+a^2)^{n-i}(a^2x+a+x-a)^{i-1} \\
&= [x^{k-1}](1+a^2)^{n-i} x^{i-1} (1+a^2)^{i-1} \\
&= [x^{k-i}](1+a^2)^{n-1} \\
&= (1+a^2)^{n-1} \delta_{ki},
\end{aligned}$$

as desired.

References

[1] R. S. Melham and Curtis Cooper. "The Eigenvectors of a Certain Matrix of binomial Coefficients." *The Fibonacci Quarterly* **38.2** (2000):123-26.

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