A recursive bijective approach to counting permutations containing 3-letter patterns

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We present a method, illustrated by several examples, to find explicit counts of permutations containing a given multiset of three letter patterns. The method is recursive, depending on bijections to reduce to the case of a smaller multiset, and involves a consideration of separate cases according to how the patterns overlap. Specifically, we use the method (i) to provide combinatorial proofs of Bóna's formula $\binom{2n-3}{n-3}$ for the number of n-permutations containing one 132 pattern and Noonan's formula $\frac{3}{n}\binom{2n}{n+3}$ for one 123 pattern, (ii) to express the number of n-permutations containing exactly k 123 patterns in terms of ballot numbers for $k \leq 4$, and (iii) to express the number of 123-avoiding n-permutations containing exactly k 132 patterns as a linear combination of powers of 2, also for $k \leq 4$. The results strengthen the conjecture that the counts are algebraic for all k.

1 Introduction

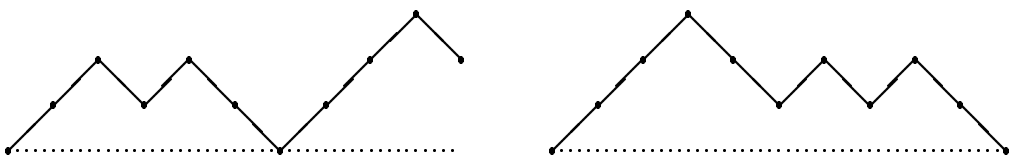
In the context of pattern-containing permutations, a *permutation* π is a list (or *word*) $\pi_1\,\pi_2\,\ldots\,\pi_n$ of distinct positive integers called letters. Viewed as a bijection, π sends its ith smallest letter to π_i. The *reverse* of a permutation $\pi_1\,\pi_2\,\ldots\,\pi_n$ is $\pi_n\,\pi_{n-1}\,\ldots\,\pi_1$ and, when the letters of π are $\{1,2,\ldots,n\}$, its *complement* is $n+1-\pi$ (termwise). The *reduced form* of a permutation π, denoted reduce(π), is obtained by replacing its smallest element by 1, its next smallest by 2, and so on. For example, reduce(9 8 2 4 6) = 5 4 1 2 3. We use $[n]$ to denote $\{1,2,\ldots,n\}$ and $[m,n]$ to denote the interval of integers $\{m,m+1,\ldots,n\}$.

A k-letter *pattern* is simply a permutation τ on $[k]$. An occurrence of τ in a permutation π is a (scattered) subword $\pi_{i_1}\,\pi_{i_2}\,\ldots\pi_{i_k}$ of π whose reduced form is τ. For example, 4 3 1 2 has two occurrences of the pattern 321, namely 4 3 1 and 4 3 2, but it is 132-avoiding. We will refer to the actual letters in an occurrence of a 321 pattern as c, b, a respectively. Similarly for 132 patterns.

Much work has recently been done counting permutations that contain a specified number of occurrences of one or more patterns. See [1] for a general survey and [2] for results on 3 letter patterns, the subject of this paper. We let $P_\tau^{(i)}(n)$ denote the set of permutations on $[n]$ containing exactly i occurrences of the pattern τ. For example, $P_{21}^{(1)}(3) = \{1\,3\,2,\ 2\,1\,3\}$. Of course, $|P_{123}^{(i)}| = |P_{321}^{(i)}|$ (reversing permutations carries patterns of one kind to the other). In section 2, we develop results counting various classes of lattice paths. These results count, via bijection, classes of 321-avoiding permutations that arise in subsequent sections.

2 Lattice Paths

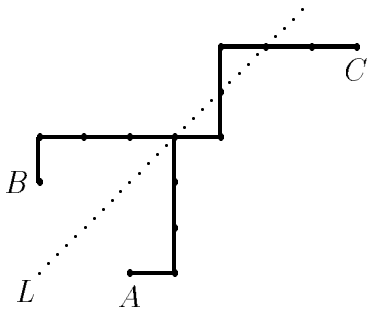
Our starting point is a known formula for counting first quadrant lattice paths of upsteps and downsteps. A *path* is a sequence of contiguous upsteps $(1,1)$ and downsteps $(1,-1)$ that starts at the origin. A *first quadrant* path is one that stays (weakly) in the first quadrant. A *balanced* path is one with an equal number of upsteps and downsteps, n, the *semilength* of the path. A *Dyck* path is a balanced first quadrant path. Illustrated are a first quadrant path of 6 upsteps and 4 downsteps and a Dyck path of semilength 4.



Both have *height* 3 and one return to the x-axis. The Dyck path's only return is at the end; it has no *interior* returns. The first path has 3 ascents (lengths of maximal sequences of contiguous upsteps) 2,1,3 and 3 descents 1,2,1. We say its *ascent sequence* is $(2,1,3)$ and its *descent sequence* is $(1,2,1)$. The ascent and descent sequences for the Dyck path shown are (3,1,1) and (2,1,2) respectively. Clearly, the requirements to be a valid ascent-descent sequence pair for a Dyck path are (i) positive integer entries, (ii) same length,

(iii) same sum, and (iv) each partial sum of the first weakly exceeds the corresponding partial sum of the second. The number of Dyck n-paths (first quadrant balanced paths of semilength n) is well known to be the Catalan number C_n and general first quadrant paths are counted by generalized Catalan numbers (ballot numbers) as follows.

Let $C(x) = \sum_{n=0}^{\infty} C_n x^n$ denote the (ordinary) generating function for the Catalan numbers and set $C_n^{(k)} = [x^n]C(x)^k$ (the ballot numbers). Thus $C_n = C_n^{(1)}$. Since multiplication of generating functions corresponds to convolution of their coefficient sequences, $C_n^{(k)}$ is the number of sequences of length k of (possibly empty) Dyck paths whose total semilength is n. By joining the k paths together with intervening upsteps—a reversible procedure—we see that $C_n^{(k)}$ is the number of first quadrant paths of $n+k-1$ upsteps and n downsteps. The marvelous André reflection principle, illustrated below, can be used to count these paths.



The number of paths from A to C that meet line L is the same as the total number of paths from B to C (reflect across the first point of contact with L). Hence, the number of paths from A to C that avoid L is the total number from A to C minus the total number from B to C. This yields $C_n^{(k)} = \binom{2n+k-1}{n} - \binom{2n+k-1}{n-1} = \frac{k}{2n+k}\binom{2n+k}{n}$. Another nice bijective proof that the number of first quadrant paths of $n+k-1$ upsteps and n downsteps is $\frac{k}{2n+k}\binom{2n+k}{n}$ is given in [6]. In brief, consider the $k\binom{2n+k}{n}$ paths of $n+k$ upsteps and n downsteps in which one of the k high points has been marked. (The high points are the leftmost points on the path at height h, $h-1$, $\ldots$, $h-k+1$ respectively, where h is the height of the path.) Then the parameter ν = x-coordinate of the marked high point is uniformly distributed over its possible values $1, 2, \ldots, 2n+k$. The kicker is that the paths with $\nu = 2n+k$ are precisely the paths that do not reach their highest level until the last upstep; deleting this step and rotating 180°, they are first quadrant paths of $n+k-1$ upsteps and n downsteps. So we have $C_n^{(k)} = \frac{k}{2n+k}\binom{2n+k}{n}$.

We will make frequent use of the convolution identity $\sum_{k=0}^{n} C_k^{(r)} C_{n-k}^{(s)} = C_n^{(r+s)}$. From the above interpretation of $C_n^{(k)}$, we have the following simple results.

Lemma 1. *The number of first quadrant paths consisting of m upsteps and n downsteps is $C_n^{(m-n+1)}$ where $C_n^{(k)} = \binom{2n+k-1}{n} - \binom{2n+k-1}{n-1} = \frac{k}{2n+k}\binom{2n+k}{n}$.*

Lemma 2. $\sum_{j=0}^{\min(m,k-m-1)} C_j^{(k-2j)} = \binom{k-1}{m} \qquad m \geq 0,\ k \geq 1$

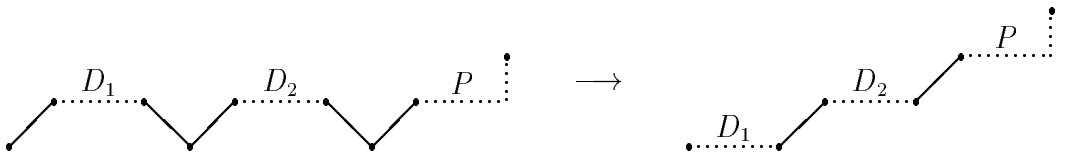
Proof. Using $C_n^{(k)} = \binom{2n+k-1}{n} - \binom{2n+k-1}{n-1}$, the left side is a telescoping sum. □

Lemma 3.

$$\begin{array}{lrcll} (i) & C_n^{(k)} - C_n^{(k-1)} & = & C_{n-1}^{(k+1)} & n,k \geq 1 \\ (ii) & C_n^{(k)} - C_{n-1}^{(k)} & = & C_n^{(k-2)} + C_{n-1}^{(k+1)} & n \geq 1,\ k \geq 2 \\ (iii) & \sum_{j=0}^{n} C_{n-j}^{(k+j)} & = & C_n^{(k+1)} & n,k \geq 0 \end{array}$$

Proof. Counting first quadrant paths by last step (up or down) yields $C_n^{(k)} = C_n^{(k-1)} + C_{n-1}^{(k+1)}$, and (i) follows. Repeating this decomposition on the first summand yields (ii). If instead we decompose the second summand and iterate, we get (iii) (after replacing k by $k+1$).

A direct combinatorial proof of (iii) can also be given. Count the $C_n^{(k+1)}$ first quadrant paths of $n+k$ upsteps and n downsteps by number of returns to the x-axis. Given such a path with $j \in [0,n]$ returns, delete the initial upstep and all j downsteps that return the path to the x-axis. For example, with $j = k = 2$ (D_1, D_2 denote Dyck paths and P denotes a first quadrant path with one more upstep than downstep),



This gives a bijection to the first quadrant paths consisting of $n+k-1$ upsteps and $n-j$ downsteps, counted by $C_{n-j}^{(k+j)}$. The inverse bijection is : insert an upstep at the start and (if $j > 0$) locate the rightmost upstep at levels $1, 2, \ldots, j$ and insert a downstep just before each of these upsteps.

A short digression: identity (iii) can be expressed as a property of the so-called Catalan

triangle $\left(C_{n-k}^{(k)}\right)_{n,k\geq 0}$:

$$\begin{array}{c|ccccccccc} n\backslash k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & & & & & & & & \\ 1 & 0 & 1 & & & & & & & \\ 2 & 0 & 1 & 1 & & & & & & \\ 3 & 0 & 2 & 2 & 1 & & & & & \\ 4 & 0 & 5 & 5 & 3 & 1 & & & & \\ 5 & 0 & 14 & 14 & 9 & 4 & 1 & & & \\ 6 & 0 & 42 & 42 & 28 & 14 & 5 & 1 & & \\ 7 & 0 & 132 & 132 & 90 & 48 & 20 & 6 & 1 & \\ 8 & 0 & 429 & 429 & 297 & 165 & 75 & 27 & 7 & 1 \end{array}$$

Pick any column. Sum the entries of each row starting in that column. The results will be the entries in the next column. The inverse of this matrix is $\left((-1)^{n-k}\binom{k}{n-k}\right)_{n,k\geq 0}$ and in fact the nonzero entries in row n are just the nonzero coefficients of $q_n(x) = x^{\frac{n}{2}} U_n(\frac{1}{2\sqrt{x}})$ where $U_n(x)$ is the Chebychev polynomial of the second kind defined by $\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta)$. This polynomial $q_n(x)$ crops up as the generating function of Dyck paths of bounded height.

Fact [7, Theorem 2] Let $u(n,h)$ denote the number of Dyck n-paths of height $\leq h$. Then

$$\sum_{n\geq 0} u(n,h)x^n = \frac{q_h(x)}{q_{h+1}(x)}.$$

This result can be generalized to count paths bounded by two given horizontal lines and ending at a given height.

Fact Let $v(n,h,r,s)$ denote the number of paths of $n+h-1$ upsteps and n downsteps that start at the origin and are weakly bounded by the line r units below the initial point (i.e. the line $y=-r$) and the line s units above the terminal point (i.e. the line $y = s+h-1$). Then

$$\sum_{n\geq 0} v(n,h,r,s)x^n = \frac{q_r(x)q_s(x)}{q_{r+s+h}(x)}.$$

Two special cases:

(i) $r = s = 0$. The generating function for paths that are bounded by two horizontal lines $h-1$ units apart, that start on the lower line and end on the upper line is $\frac{1}{q_h(x)}$.

(ii) $r = h-1$, $s = 0$. The generating function for paths that are bounded by two horizontal lines $2(h-1)$ units apart, that start midway between the lines and end on the upper line is $\frac{q_{h-1}(x)}{q_{2h-1}(x)} = \frac{1}{p_h(x)}$ where $p_h(x) = 2x^{\frac{n}{2}} T_h(\frac{1}{2\sqrt{x}})$ and T_n, defined by $T_n(\cos\theta) = \cos n\theta$, is the Chebychev polynomial of the first kind.

Thus both kinds of Chebychev polynomials have lattice path interpretations. End of digression.

The next few results count various classes of Dyck paths to serve as the induction basis for recursively counting pattern-containing permutations.

Lemma 4. *The number of Dyck n-paths with first ascent $= k$ is $C_{n-k}^{(k)}$ and the number with first ascent $\geq k$ is $C_{n-k}^{(k+1)}$.*

Proof. A path with first ascent $= k$ begins with k upsteps followed by a downstep. Deleting these k upsteps and 1 downstep and reading the resulting path backwards gives a bijection to first quadrant paths of $n-1$ upsteps and $n-k$ downsteps, counted by $C_{n-k}^{(k)}$. For paths with first ascent $\geq k$, delete the first k upsteps and read backwards to get a bijection to first quadrant paths of n upsteps and $n-k$ downsteps, counted by $C_{n-k}^{(k+1)}$. □

Lemma 5. *(i) The number of Dyck n-paths with first ascent $\geq r$, at least one interior return, and last descent $\geq s$ is $C_{n-r-s}^{(r+s+1)}$.*
(ii) The total number of Dyck n-paths with first ascent $\geq r$ and last descent $\geq s$ is $\sum_{j=0}^{\min(r,s)} C_{n-r-s+j}^{(r+s+1-2j)}$

Proof. (i) Given such a path, remove the first downstep that returns the path to the x-axis and transfer the first upstep to the vacated location, then place the removed downstep at the end of the path. Repeat this process r times altogether. This gives a bijection to Dyck n-paths whose last descent is $\geq r+s$ and, reading the paths backwards, to Dyck n-paths whose first ascent is $\geq r+s$, counted by $C_{n-r-s}^{(r+s+1)}$ (Lemma 4).
(ii) Using (i), count by the lowest point on the path. □

Lemma 6. *For $a, d, i \geq 1$ and $n \geq i+d$, the Dyck n-paths with first ascent a, last descent*

d, and first i nonfinal *descents all* $= 1$ *are equinumerous with the Dyck n-paths with first ascent* $a + i$ *and last descent* d.

Proof. There is a cut-and-paste bijection from the first set to the second: simply transfer the upstep immediately following each of the first i downsteps to the start of the path. It's inverse is: remove i upsteps from the start of the path, and insert them, one apiece, immediately after each of the first i downsteps (which will not be steps of the final descent because $n \geq i + d$). □

Lemma 7. *The Dyck n-paths with first ascent* $\geq r$ *and first s nonfinal descents all* $= 1$ *are equinumerous with the Dyck n-paths with first ascent* $\geq r + s$*, and hence counted by* $C_{n-r-s}^{(r+s+1)}$.

Proof. Sum the previous lemma's count over $a \geq r$ and $d \geq 1$ with $i := s$. □

Lemma 8. *The number of Dyck n-paths with first ascent* $\geq r$ *and last* $s - 1$ *noninitial ascents all* $= 1$ *is*

$$\sum_{j=0}^{m} C_{n-r-s+j}^{(r+s+1-2j)} - \sum_{j=2}^{m} \binom{r+s-2j}{r-j} C_{n-r-s+j}^{(0)}$$

where $m = \min(r, s)$.

Proof. Reading the paths backwards, they correspond to Dyck n-paths with first $s - 1$ nonfinal descents all $= 1$ and last descent $\geq r$. For $n \geq r + s - 1$, the latter correspond by Lemma 6 to Dyck n-paths with first ascent $\geq s$ and last descent $\geq r$ and hence are counted by the first sum (Lemma 5 (ii)). The second sum is a correction for the case $n < r + s - 1$ (using Lemma 2). □

There is another formulation of the previous result that avoids subtraction:

Lemma 9. *The number of Dyck n-paths with first ascent* $\geq r$ *and last* $s - 1$ *noninitial ascents all* $= 1$ *is*

$$C_{n+1-r-s}^{(r+s)} + \sum_{k=0}^{r+s-4} \sum_{j=0}^{r+s-4-k} \binom{k}{r-2-j} C_{n-r-s+1}^{(r+s-2-k)}.$$

Proof. Let $F(r,s) = \sum_{n\geq 0}\Big(C^{(r+s)}_{n+1-r-s} + \sum_{k=0}^{r+s-4}\ \sum_{j=0}^{r+s-4-k}\binom{k}{r-2-j}C^{(r+s-2-k)}_{n+1-r-s}\Big)x^n$ and $G(r,s) = \sum_{n\geq 0}\Big(\sum_{j=0}^{\min(r,s)} C^{(r+s+1-2j)}_{n-r-s+j} - \sum_{j=2}^{\min(r,s)}\binom{r+s-2j}{r-j}C^{(0)}_{n-r-s+j}\Big)x^n$ denote the generating functions for the two expressions whose equality we wish to establish. We will show that F and G both satisfy the recurrence

$$u(r,s) = xu(r-1,s) + xu(r,s-1) + x^{r+s-1}C(x)^{r+s-2}\big(1-C(x)\big)^2 \qquad r,s\geq 2 \qquad (1)$$

with initial condition

$$u(r,1) = u(1,r) = x^r C(x)^{r+1} \qquad r\geq 1$$

(and hence $F(r,s) = G(r,s)$ for all $r,s\geq 1$).

From the definition of $C^{(m)}_n$, we have

$$F(r,s) = x^{r+s-1}C(x)^{r+s} + \sum_{k=0}^{r+s-4}\ \sum_{j=0}^{r+s-4-k}\binom{k}{r-2-j}x^{r+s-1}C(x)^{r+s-2-k}$$

and

$$G(r,s) = \sum_{j=0}^{\min(r,s)} x^{r+s-j}C(x)^{r+s+1-2j} - \sum_{j=2}^{\min(r,s)}\binom{r+s-2j}{r-j}x^{r+s-j}.$$

It is routine to verify $G(r,1) = x^rC(x)^{r+1}$ using the basic identity $xC(x)^2 = C(x)-1$ for the Catalan numbers generating function. Since both G and the recurrence are symmetric in r,s it suffices to verify that G satisfies the recurrence for $r\geq s$ (≥ 2). First suppose $r>s$; this guarantees $\min(r-1,s) = s$. (With a little care, detailed later, the same calculation works in case $r=s$.) Adjusting summations to terminate at s and using the basic identity $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$, integer $a,b\geq 0$, the right hand side of (1) with $u=G$ can be written

$$2\sum_{j=0}^{s} x^{r+j}C(x)^{r-s+2j} - x^rC(x)^{r-s} - \sum_{j=2}^{s}\binom{r+s-2j}{r-j}x^{r+s-j} + x^{r+s-1}C(x)^{r+s-2}\big(1-C(x)\big)^2.$$

The left hand side is $G(r,s)$, and the difference is

$$x^rC(x)^{r-s}\big(2-C(x)\big)\sum_{j=0}^{s}\big(xC(x)^2\big)^j - x^rC(x)^{r-s} + x^{r+s-1}C(x)^{r+s-2}\big(1-C(x)\big)^2.$$

Evaluating the sum and using $2-C(x) = 1-xC(x)^2$, this difference simplifies to $x^{r+s-1}C(x)^{r+s-2}\Big(\big(1-C(x)\big)^2 - x^2C(x)^4\Big) = 0$.

In case $r = s$, the two sums in $G(r-1,s)$ terminate at $s-1$ rather than s, but the contributions from $j = s$ cancel provided we define $\binom{-1}{-1} = 1$. The adjustment of summations to end at s holds provided $\binom{-1}{0} = 0$. Making these definitions, the identity $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$ holds also for $a = -1$, $b = 0$ and the entire calculation above goes through for $r = s$.

As for F, reversing the inner summation and then replacing the lower binomial coefficient parameter by its complement shows that F is symmetric in r and s. We have $F(r,1) = x^r C(x)^{r+1}$ because the double sum makes no contribution when $s = 1$. The right hand side of (1) with $u = F$ is, after combining terms,

$$\begin{aligned}
& x^{r+s-1}\Big(C(x)^{r+s} + C(x)^{r+s-2} + \sum_{\substack{j,k\geq 0,\\ j+k\leq r+s-5}} \binom{k+1}{r-2-j} C(x)^{r+s-k-3}\Big) \\
= \;& x^{r+s-1}\Big(C(x)^{r+s} + C(x)^{r+s-2} + \sum_{\substack{j\geq 0, K\geq 1,\\ j+K\leq r+s-4}} \binom{K}{r-2-j} C(x)^{r+s-K-2}\Big) \\
= \;& x^{r+s-1}\Big(C(x)^{r+s} + \sum_{\substack{j,K\geq 0\\ j+K\leq r+s-4}} \binom{K}{r-2-j} C(x)^{r+s-K-2}\Big) \\
= \;& F(r,s) \quad = \quad \text{the left hand side of (1).}
\end{aligned}$$

(The hypothesis $r, s \geq 2$ is needed for the next to last equality.)

□

Lemma 10.

$$\sum_{k=0}^{m-1} \binom{2m-2k}{m-1-k} C_k = \binom{2m+1}{m-1}.$$

Proof. The right side counts lattice paths of $m+2$ upsteps and $m-1$ downsteps. Given such a path P, note that it ends at height 3 and split it into two subpaths P_1, P_2 by deleting the last upstep that carries P to height 3. Then P_1 is a path with 2 more upsteps than downsteps and P_2 is a Dyck path. Counting by the semilength k of this Dyck path yields the claimed identity. □

3 321-avoiding permutations

Several bijections are known [7, 8] among $P_{321}^{(0)}(n)$, $P_{132}^{(0)}(n)$ and Dyck paths of semilength n, proving that $|P_{321}^{(0)}(n)| = |P_{132}^{(0)}(n)| = C_n$. We need to examine one of them. Define the difference operator Δ on a finite sequence, as usual, by $\Delta\big((u_i)_{i=1}^n\big) = (u_i - u_{i-1})_{i=2}^n$.

Theorem 11 (Krattenthaler[7]). *There is a bijection $\pi \to P$ from $P_{321}^{(0)}(n)$ to Dyck n-paths that sends π_1 to the first ascent of P.*

The bijection can be described as follows (see [2] for a nice pictorial description). Given $\pi \in P_{321}^{(0)}(n)$, let $1 = i_1 < i_2 < \ldots < i_k$ denote the positions (locations) of the successive record highs (left to right maxima) in π. Then $\Delta(0, \pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k} = n)$ is the ascent sequence of P and $\Delta(i_1 = 1, i_2, \ldots, i_k, n+1)$ is the descent sequence of P. For example, $2\,1\,4\,7\,3\,5\,6 \in P_{321}^{(0)}(7)$ has record highs of 2,4,7 in positions 1,3,4 yielding ascent sequence $\Delta(0, 2, 4, 7) = (2, 2, 3)$ and descent sequence $\Delta(1, 3, 4, 8) = (2, 1, 4)$. Since $i_1 = 1$, we see that π_1 is the first ascent of P.

We use this bijection to count certain classes of 321-avoiding permutations.

Lemma 12. $\#\{\pi \in P_{321}^{(0)}(n) : \ \pi_1 = k\} = C_{n-k}^{(k)}$ *and* $\#\{\pi \in P_{321}^{(0)}(n) : \ \pi_1 \geq k\} = C_{n-k}^{(k+1)}$

Proof. Immediate from the bijection and Lemma 4. □

Lemma 13. *For $m \in [n]$, the 321-avoiding permutations on $[n]$ with the following properties are equinumerous and all are counted by $C_{n-m}^{(m+1)}$.*

1. *1 occurs no earlier than position m*
2. *first entry is $\geq m$*
3. *n occurs no later than the mth position from the end*
4. *last entry is $\leq n + 1 - m$*

Proof. The map "reverse and complement" is a bijection from the first set to the third set and also from the second to the fourth. For a permutation π in the first set, the first $m-1$ entries are increasing and the first entry is ≥ 2. So the table of record highs and their locations begins $\begin{smallmatrix} \text{location:} & 1 & 2 & \ldots & m-1 \\ \text{record high:} & \pi_1 & \pi_2 & \ldots & \pi_{m-1} \end{smallmatrix}$ with $\pi_1 \geq 2$. Under Krattenthaler's bijection, the descent sequence of the corresponding Dyck path, $\Delta(1, 2, \ldots, m-1, \textit{other locations})$,

begins $\underbrace{1,1,\ldots,1}_{m-2}$ and its ascent sequence, $\Delta(0,\pi_1,\ldots)$, begins with $\pi_1 \geq 2$. This means we get all Dyck paths with first ascent ≥ 2 and the first $m-2$ descents all $= 1$. The second set is sent to Dyck n-paths with first ascent $\geq m$. Lemma 7 now yields the result. □

Lemma 14. *Let $n > i \geq 1$, $k \in [n]$ and consider the 321-avoiding permutations on $[n]$ with first entry k. Those with last i entries increasing in value, $\{\pi \in P^{(0)}_{321}(n) :\ \pi_1 = k,$ last i $\uparrow\}$, are equinumerous with those in which n does not occur among the last $i-1$ entries, $\{\pi \in P^{(0)}_{321}(n) :\ \pi_1 = k,\ n = \pi_j$ with $j \leq n-(i-1)\}$.*

Proof. We define a bijection ϕ from the first set to the second: ϕ is the identity if $n = \pi_j$ with $j \leq n-i$, otherwise, the hypothesis "last i $\uparrow$" implies $n = \pi_n$ and ϕ interchanges π_n and π_{n-i+1}. □

Lemma 15.

$$\#\{\pi \in P^{(0)}_{321}(n) :\ \pi_1 = k,\ n \text{ does not occur among the last } i-1 \text{ entries}\} =$$
$$\#\{\text{Dyck } n\text{-paths: first ascent} \geq k \text{ and last descent} \geq i\}.$$

Proof. Krattenthaler's bijection sends the first set to the second. This is because $\pi \in P^{(0)}_{321}(n)$ does not have n among the last $i-1$ entries $\Leftrightarrow$ the last location of a record high in π is $\leq n-(i-1) \Leftrightarrow$ the last entry of $\Delta(\textit{record high positions}, n+1)$ is $\geq i \Leftrightarrow$ the last descent of the corresponding Dyck path is $\geq i$. □

In particular, using Lemmas 3 (i) and 5 (ii), $\#\{\pi \in P^{(0)}_{321}(n) : \text{first entry} \geq 2 \text{ and last entry} \leq n-1\} = C^{(1)}_{n-2} + C^{(3)}_{n-3} + C^{(5)}_{n-4} = C^{(2)}_{n-2} + C^{(5)}_{n-4}$.

4 One 132 Pattern

As a warmup, we treat this case in full detail.

Lemma 16. *Suppose $\pi \in P^{(1)}_{132}(n)$ and $a\,c\,b$ is the unique 132 pattern in π. Then a, c are consecutive in position in π and a, b are consecutive in value (i.e. $b = a+1$).*

Proof. Suppose some letter, i say, appears between a and c in π so that $\pi = \cdots a \cdots i \cdots c \cdots b \cdots$. If $i > b$, then $a\,i\,b$ is another 132 pattern in π and if $i < b$, then $i\,c\,b$ is another 132 pattern. Thus there are no possibilities for i and a, c are consecutive in position. Now suppose i

lies between a and b in value: $a < i < b$. If i occurs before c in π, then $i\,c\,b$ is a 132, while if i occurs after c, then $a\,c\,i$ is a 132. Again there are no possibilities for i and $b = a + 1$. □

Lemma 17. $\#\{\pi \in P_{132}^{(1)}(n) : \textit{the 132 pattern occupies consecutive positions in } \pi\} = (n-2)C_{n-2} = \binom{2n-4}{n-3}$.

Proof. Given $\pi \in P_{132}^{(1)}(n)$ with a, c, b in positions k, $k+1$, $k+2$, write π as $W_1\,a\,c\,b\,W_2$. All letters in W_1 are $> b$ (if $x < b$, $x \in W_1$, then $x\,c\,b$ is a 132) and no letter in W_2 lies between b and c in value (if $b < x < c$, $x \in W_2$, then $a\,c\,x$ is a 132).

We now claim the map $\pi \to (\rho, k)$ with $\rho = \text{reduce}(W_1\,c\,W_2)$ is a bijection to $P_{132}^{(0)}(n-2) \times [n-2]$; since $|P_{132}^{(0)}(n)| = C_n$, the lemma will follow. To form ρ, a and b will be deleted from π and 2 subtracted from all letters $> b$. So ρ will have the form $W_1'\,c'\,W_2'$ with $W_1' = W_1 - 2$ (entrywise), $c' = c - 2$ in position k, and the number of successors of c' in ρ that are $< c'$ will be precisely a. This permits a and hence $b = a + 1$ to be recovered from (ρ, k) and so π can be reconstructed by adding 2 to each letter of ρ that is $> a$ and inserting a, b so they occupy positions k and $k+2$. □

Lemma 18. $\#\{\pi \in P_{132}^{(1)}(n) : \textit{the 132 pattern occupies the first, second and last positions in } \pi\} = C_{n-3}$.

Proof. Here π has the form $a\,c\,W_2\,b$ and, necessarily, $(a, c, b) = (n-2, n, n-1)$ (else more than one 132 pattern). Hence the map $\pi \to W_2$ is a bijection to the set $P_{132}^{(0)}(n-3)$, counted by C_{n-3}. □

Now we count $P_{132}^{(1)}(n)$ by the number k of letters between the c and b of the 132 pattern.

Proposition 19. *For $k \in [0, n-3]$, there is a bijection $\phi : \{\pi \in P_{132}^{(1)}(n) : \textit{the } c \textit{ and } b \textit{ of the 132 pattern are separated by } k \textit{ other letters}\} \to \{\rho \in P_{132}^{(1)}(n-k) : \textit{ the 132 pattern of } \rho \textit{ occupies consecutive positions in } \rho\} \times \{\sigma \in P_{132}^{(1)}(k+3) : \textit{ the 132 pattern of } \sigma \textit{ occupies the first, second and last positions in } \sigma\}$.*

Proof. Here π has the form $W_1\,a\,c\,W_2\,b\,W_3$ with k letters in the subword W_2. In fact, $k < a$ and the letters in W_2 comprise $\{a-1, a-2, \ldots, a-k\}$, else too many 132's. Also, and for the same reason, the letters $1, 2, \ldots, a-k-1$ all occur in W_3. These facts

imply the map $\phi : \ \pi \to (\rho, \sigma)$ with $\rho = \text{reduce}(W_1\, a\, c\, b\, W_3) \in P^{(1)}_{132}(n-k)$ and $\sigma = \text{reduce}(a\, c\, W_2, b) \in P^{(1)}_{132}(k+3)$ is the desired bijection. This is because ρ is obtained from $(W_1\, a\, c\, b\, W_3)$ by subtracting k from all letters $\geq a$ and $\sigma = k{+}1\ k{+}3\ W_2{-}(a{-}k{-}1)\ k{+}2$. Hence the 132 pattern in ρ identifies the values of a, b, c and then knowing a, W_1 and W_3 can be recovered from ρ as can W_2 from σ. □

Hence

$$\begin{aligned}
|P^{(1)}_{132}(n)| &= \sum_{k=0}^{n-3} \#\{\rho \in P^{(1)}_{132}(n-k) : \ a,\, c,\, b \text{ occupy consecutive positions in } \rho\} \times \\
&\qquad \#\{\sigma \in P^{(1)}_{132}(k+3) : \ a,\, c,\, b \text{ occupy the first, second and last positions in } \sigma\} \\
&\overset{(1)}{=} \sum_{k=0}^{n-3} \binom{2(n-k)-4}{(n-k)-3} \times C_{(k+3)-3} \\
&= \sum_{k=0}^{n-3} \binom{2n-2k-4}{n-k-3} C_k \\
&= \binom{2n-3}{n-3},
\end{aligned}$$

(1) by Lemmas 17 and 18, the last equality by Lemma 10. We have established

Theorem 20 (Bóna [4]). $|P^{(1)}_{132}(n)| = \binom{2n-3}{n-3}$.

5 One 321 Pattern

We wish to show bijectively that $|P^{(1)}_{321}(n)| = C^{(6)}_{n-3}$ $\left(= \frac{3}{n}\binom{2n}{n-3}\right)$ for $n \geq 3$.

Lemma 21. *Suppose the* 321 *pattern in* $\pi \in P^{(1)}_{321}(n)$ *has middle letter* b. *Then the letters preceding* b *in* π *consist of the* $b-1$ *letters* $\{c\} \cup [b-1] \setminus \{a\}$ (*and so* b *is a fixed point*).

Proof. The letters preceding b in π include c but no other letter $> b$ (else more than one 321 pattern). Among the letters following b, a is the only one $< b$ (else again more than one 321 pattern). The result follows. □

Proposition 22. *Let* $b \in [2, n-1]$. *Then there is a bijection from* $\{\pi \in P^{(1)}_{321}(n) : \text{middle letter of } 321 \text{ pattern is } b\} \longrightarrow \{\rho \in P^{(0)}_{321}(b) : \text{last entry} \leq b{-}1\} \times \{\sigma \in P^{(0)}_{321}(n{-}b{+}1) : \text{first entry} \geq 2\}$.

Proof. Write π as $W_1\, b\, W_2$ with c occurring in the subword W_1 and a in W_2. Set $\rho = \text{reduce}(W_1\, a)$ and $\sigma = \text{reduce}(c\, W_2)$. Note that ρ is simply $(W_1' a)$ where W_1' is W_1 with c replaced by b and, since $\{c, W_2\}$ consists of all letters $> b$ together with a, σ is obtained from $c\, W_2$ by replacing a by 1, and subtracting $b-1$ from every other entry. It is clear that ρ, σ have the specified properties. Also the last letter of ρ is a and ρ contains all information about W_1 except the value of c. On the other hand, σ contains the value of c (via its first letter) and all information about W_2 except the value of a. Thus we can uniquely recover π from ρ and σ and the map $\pi \to (\rho, \sigma)$ is a bijection as claimed.

□

The sets in the Cartesian product of the preceding Proposition are counted by $C^{(3)}_{b-2}$ and $C^{(3)}_{n-b-1}$ respectively (Lemma 12). Hence, counting by the middle letter of the 321 pattern, we find

$$|P^{(1)}_{321}(n)| = \sum_{b=2}^{n-1} C^{(3)}_{b-2} C^{(3)}_{n-b-1} = \sum_{b=0}^{n-3} C^{(3)}_{b} C^{(3)}_{n-3-b} = C^{(6)}_{n-3}$$

and we obtain

Theorem 23 (Noonan [3]). $|P^{(1)}_{321}(n)| = C^{(6)}_{n-3}$.

It is observed in [3] that the number of permutations in $P^{(0)}_{321}(n+2)$ in which $1, 2, 3, 4, 5$ occur in increasing order is also $C^{(6)}_{n-3}$. This can be explained as follows: incrementing by 1 all entries ≥ 6 in such a permutation and inserting the letter 6 at the start gives a bijection to $\{\pi \in P^{(0)}_{321}(n+3) : \ \pi_1 = 6\}$, counted by $C^{(6)}_{n-3}$ (Lemma 12).

6 Two 321 Patterns

For $P^{(2)}_{321}(n)$ we distinguish two cases according as the two 321 patterns have (I) a common (i.e. the same) middle letter b or (II) distinct middle letters. Case I is counted by b itself, case II by the number of intervening letters.

Case I: Same middle letter. Here, the two 321 patterns also have a common first letter or a common last letter (or there would be at least four 321 patterns) and by symmetry there are the same number of each type. We count the second type, with patterns $c_1 ba$ and $c_2 ba$ so that π can be written $W_1 b W_2$ with c_1, c_2 occurring in W_1 and a in W_2.

Lemma 24. *For π as just described, the subword $W_1 a$ consists of the letters $[b-1] \cup \{c_1, c_2\}$ and $c_1 c_2 W_2$ consists of $\{a\} \cup [b+1, n]$. Consequently, $|W_1| = b$ and b is in position $b+1 \in [3, n-1]$.*

Proof. Similar to that of Lemma 15: anything else would give too many 321 patterns. □

Proposition 25. *Let $b \in [2, n-2]$. Then there is a bijection from $\{\pi \in P_{321}^{(2)}(n)$: the two 321 patterns in π have b as common middle letter and also a common last letter$\}$ to $\{\rho \in P_{321}^{(0)}(b+1)$: last entry $\leq$ largest entry $(= b+1) - 2\} \times \{\sigma \in P_{321}^{(0)}(n-b+1)$: the letter 1 occurs at or after position 3$\}$.*

Proof. Write $\pi = W_1\, b\, W_2$ with c_1, c_2 in W_1 and the common a in W_2. Set $\rho = \text{reduce}(W_1\, a)$ and $\sigma = \text{reduce}(c_1\, c_2\, W_2)$. The proof that ρ, σ range over the specified sets and contain enough information to recover π is much like that of Proposition 22. □

Corollary 26. $\#\{\pi \in P_{321}^{(2)}(n)$: the two 321 patterns in π have a common middle letter$\} = 2C_{n-4}^{(8)}$.

Proof. Assume a common last letter (as noted above, the required count is double this count). The sets in the Cartesian product have sizes $C_{b-2}^{(4)}$ and $C_{n-b-2}^{(4)}$ respectively (Lemma 8). Summing over $b \in [2, n-2]$ gives the result. □

Case II: Distinct middle letters. Let $c_1\, b_1\, a_1$ and $c_2\, b_2\, a_2$ be the two 321 patterns with $c_1 \leq c_2$ and write π as $W_1\, b_1\, W_2\, b_2\, W_3$. Clearly, c_1 occurs in W_1, and c_2 in W_2 unless $c_1 = c_2$. Also, a_2 occurs in W_3, and a_1 in W_2 unless $a_1 = a_2$. This seems to give three cases (four if you distinguish the order c_2 and a_1 occur in W_2) but we can handle them all together. In order not to exceed the quota of 321 patterns, the subword W_1 consists of the letters $[b_1 - 1] \cup \{c_1\} \backslash \{a_1\}$, W_2 consists of $[b_1 + 1, b_2 - 1] \cup \{a_1, c_2\} \backslash \{c_1, a_2\}$ (in the multiset sense, since a_1 may $= a_2$ or c_1 may $= c_2$), and W_3 consists of $[b_2 + 1, n] \cup \{a_2\} \backslash \{c_2\}$. In any case $|W_1| = b_1 - 1$, $|W_2| = b_2 - b_1 - 1$, $|W_3| = n - b_2$.

Proposition 27. *Let $k \in [0, n-4]$. Then there is a bijection from $\{\pi \in P_{321}^{(2)}(n)$: the two 321 patterns in π have distinct middle letters separated by k other letters$\}$ to $\{\rho \in P_{321}^{(1)}(n-k-1)\} \times \{\sigma \in P_{321}^{(0)}(k+2)$: first entry of $\sigma \geq$ smallest entry $+1$, last entry of $\sigma \leq$ largest entry $-1\}$.*

Proof. Write $\pi = W_1\, b_1\, W_2\, b_2\, W_3$. Set $\rho = \text{reduce}(W_1'\, b_1\, W_3')$ where W_1' is W_1 with c_1 replaced by c_2 (this includes the case $c_1 = c_2$ in which case $W_1' = W_1$) and W_3' is W_3 with a_2 replaced by a_1. Set $\sigma = \text{reduce}(c_1\, W_2\, a_2)$. Then ρ, σ lie in the specified sets.

To get ρ from $W_1'\, b_1\, W_3'$, all letters $\leq b_1$ are left intact and $b_2 - b_1$ is subtracted from all letters $> b_2$ (there are no letters in $[b_1, b_2 - 1]$.) Now W_1 (and W_1') has only one letter $> b_1$ and so the position of c_1 in π is the position of the "c" in the 321 pattern of ρ. Adding $b_2 - b_1$ to this "c" gives the value of c_2. The "a" of the pattern is a_2. The other letters of ρ yield both position and value of all letters in W_1 and W_3 other than c_1 and a_2.

To see what σ is, note that $c_1\, W_2\, a_2$ consists of $\{a_1\} \cup [b_1 + 1, b_2 - 1] \cup \{c_2\}$. Hence reducing will send a_1 to 1, c_2 to $k + 2$ (since there are k letters in $[b_1 + 1, b_2 - 1]$), and all other letters will be decremented by $b_1 - 1$. So the position of 1 in σ yields the position of a_1 in $c_1\, W_2\, a_2$ (if 1 is in the last position, then $a_1 = a_2$). Likewise, the position of $k + 2$ in σ yields the position of c_2 in $c_1\, W_2\, a_2$ (if $k + 2$ is the first letter, then $c_2 = c_1$). Adding $b_1 - 1$ to the other letters in σ yields positions and values of all letters in $c_1\, W_2\, a_2$ other than c_2 and a_1.

Summarizing, ρ gives the nonpattern letters in W_1 and W_3, σ those in W_2; ρ gives the position of c_1 and a_2 in π and the values of c_2 and a_1 while σ gives the positions of a_1 and c_2, tells if $a_1 = a_2$ or $c_2 = c_1$ and if not, gives the values of a_2 and c_1. So ρ and σ contain just enough information to recover π and the map $\pi \to (\rho, \sigma)$ is a bijection as claimed. □

Corollary 28. $\#\{\pi \in P_{321}^{(2)}(n) :$ *the two* 321 *patterns in* π *have distinct middle letters*$\} = C_{n-4}^{(8)} + C_6^{(11)}$.

Proof. The sizes of the sets in the Cartesian product of the Proposition are $C_{n-k-4}^{(6)}$ (Theorem 23) and $C_k^{(2)} + C_{k-2}^{(5)}$ (Lemma 15) and summing over $k \in [0, n-4]$ gives the result. □

Combining Cases I and II, we have a result conjectured in [5] and proved in [2].

Theorem 29. $|P_{321}^{(2)}(n)| = 3C_{n-4}^{(8)} + C_{n-6}^{(11)}$.

7 Three and Four 321 Patterns

For these situations, counts are also needed of subsets of $P_{321}^{(1)}$ and $P_{321}^{(2)}$ satisfying certain conditions on the values of the first/last letters and/or the positions of the max/min letters. The computations are laid out in the following tables (considerable details are left to the reader). The expressions obtained are by no means unique. For example, the total in the first table below, $2C_{n-4}^{(6)} + C_{n-6}^{(9)}$ could also be rendered $C_{n-4}^{(5)} + C_{n-4}^{(7)}$ using Lemma 3. Denote the 321 patterns $c_1b_1a_1, c_1b_1a_1, \ldots$ ordered primarily by increasing b, secondarily by increasing c and finally by increasing a. In the tables, $\Delta C_n^{(k)}$ means $C_n^{(k)} - C_{n-1}^{(k)}$. The counts in Cases 2 and 3 for three 321 patterns need to be doubled due to symmetry considerations. Likewise for Cases 2, 4 and 5 for four 321 patterns.

$\{\pi \in P_{321}^{(1)}(n) : \text{ last } 2 \uparrow\}$

Case	there is only one case
pattern type	$\pi = W_1\, b\, W_2$
position of b	b
$\rho =$	reduce$(W_1 a)$
ranges over	$P_{321}^{(0)}(b)$
subject to	last $\neq$ max
a set of size	$C_{b-2}^{(3)}$
$\sigma =$	reduce(cW_2)
ranges over	$P_{321}^{(0)}(n-b+1)$
subject to	first ≥ 2, last $2 \uparrow$
a set of size	$2C_{n-b-2}^{(3)} + C_{n-b-4}^{(6)}$
sum over	$b \in [2, n-2]$
total	$2C_{n-4}^{(6)} + C_{n-6}^{(9)}$

$\{\pi \in P_{321}^{(2)}(n) :$ last 2 $\uparrow\}$

Case	1	2	3
pattern type	distinct b's $\pi = W_1\, b_1\, \underbrace{W_2}_{k}\, b_2 W_3$	common b's common a's	common b's common c's
position of b	both are fixed points	$b+1$	$b-1$
$\rho =$	reduce$(W_1' b_1 W_3')$ $W_1' = W_1/.\, c_1 \leftarrow c_2$ $W_3' = W_3/.\, a_2 \leftarrow a_1$	reduce$(W_1 a)$	reduce$(W_1 a_1 a_2)$
ranges over	$P_{321}^{(1)}(n-k-1)$	$P_{321}^{(0)}(b+1)$	$P_{321}^{(0)}(b)$
subject to	last 2 $\uparrow$	last $\leq$ max-2	max in pos ≤ -3
a set of size	$2C_{n-k-5}^{(6)} + C_{n-k-7}^{(9)}$	$C_{b-2}^{(4)}$	$C_{b-3}^{(4)}$
$\sigma =$	reduce$(c_1 W_2 a_2)$	reduce$(c_1 c_2 W_2)$	reduce$(c\, W_2)$
ranges over	$P_{321}^{(0)}(k+2)$	$P_{321}^{(0)}(n-b+1)$	$P_{321}^{(0)}(n-b+2)$
subject to	first $>$ min, last $<$ max	1 weakly after pos 3 and last 2 $\uparrow$	first ≥ 3 and last 2 $\uparrow$
a set of size	$C_k^{(2)} + C_{k-2}^{(5)}$	$C_{n-b-4}^{(6)} + C_{n-b-3}^{(4)} +$ $C_{n-b-2}^{(2)} - C_{n-b-2}^{(0)}$	$C_{n-b-3}^{(6)} + C_{n-b-2}^{(4)}$ $+C_{n-b-1}^{(2)}$
sum over	$k \in [0, n-5]$	$b \in [2, n-3]$	$b \in [3, n-1]$
case total	$2C_{n-5}^{(8)} + 3C_{n-7}^{(11)} + C_{n-9}^{(14)}$	$C_{n-6}^{(10)} + C_{n-5}^{(8)}$ $+C_{n-4}^{(6)} - C_{n-4}^{(4)}$	$C_{n-6}^{(10)} + C_{n-5}^{(8)} + C_{n-4}^{(6)}$

Grand Total: $4C_{n-5}^{(8)} + 3C_{n-7}^{(11)} + C_{n-9}^{(14)} + 2C_{n-6}^{(10)} + 2C_{n-4}^{(6)} - C_{n-4}^{(4)}$

Three 321 patterns

$\pi = W_1\, b\, W_2$

Case	1	2	3
pattern type	$b_1 \le b_2 < b_3 \equiv b$	$b_1 < b_2 = b_3 \equiv b,\ c_2 = c_3 \equiv c$	common b's and a's
position of b	b	$b-1$	$b+2$
$\rho =$	reduce$(W_1 a_3)$	reduce$(W_1 a_2 a_3)$	reduce$(W_1 a)$
ranges over	$P_{321}^{(2)}(b)$	$P_{321}^{(1)}(b)$	$P_{321}^{(0)}(b+2)$
subject to	last $\neq$ max	last two $\uparrow$ and max occurs earlier	last $\le$ max-3
a set of size	$\Delta P_{321}^{(2)}(b)$	$C_{b-4}^{(6)} + C_{b-6}^{(9)}$	$C_{b-2}^{(5)}$
$\sigma =$	reduce$(c_3 W_2)$	reduce$(c W_2)$	reduce$(c_1 c_2 c_3 W_2)$
ranges over	$P_{321}^{(0)}(n-b+1)$	$P_{321}^{(0)}(n-b+2)$	$P_{321}^{(0)}(n-b+1)$
subject to	$\sigma_1 \ge 2$	$\sigma_1 \ge 3$	1 weakly after pos 4
a set of size	$C_{n-b-1}^{(3)}$	$C_{n-b-1}^{(4)}$	$C_{n-b-3}^{(5)}$
sum over	$b \in [4, n-1]$	$b \in [4, n-1]$	$b \in [2, n-3]$
case total	$\Delta(3C_{n-5}^{(11)} + C_{n-7}^{(14)})$	$C_{n-5}^{(10)} + C_{n-7}^{(13)}$	$C_{n-5}^{(10)}$

Grand Total: $7C_{n-5}^{(10)} + 6C_{n-7}^{(13)} + C_{n-9}^{(16)}$

Four 321 patterns

Case	1	2	3
pattern type	$b_3 < b_4 \equiv b$	$b_2 < b_3 = b_4 \equiv b,\ c_3 = c_4 \equiv c$	4321 pattern present
position of b	b	$b-1$	c, b adjacent, $c = b+1$ $\pi = W_1\, c\, b\, W_2$
$\rho =$	reduce$(W_1 a_4)$	reduce$(W_1 a_3 a_4)$	$W_1 c W_2$
ranges over	$P_{321}^{(3)}(b)$	$P_{321}^{(2)}(b)$	$P_{321}^{(1)}(n-1)$
subject to	last $\neq$ max	last two $\uparrow$ and max occurs earlier	—
a set of size	$\Delta P_{321}^{(3)}(b)$	$C_{b-5}^{(8)} + 2C_{b-7}^{(11)} + C_{b-9}^{(14)} +$ $2C_{b-6}^{(10)} + 2C_{b-4}^{(6)} - C_{b-4}^{(4)}$	$C_{b-4}^{(6)}$
$\sigma =$	reduce$(c_4 W_2)$	reduce(cW_2)	not needed
ranges over	$P_{321}^{(0)}(n-b+1)$	$P_{321}^{(0)}(n-b+2)$	
subject to	$\sigma_1 \geq 2$	$\sigma_1 \geq 3$	
a set of size	$C_{n-b-1}^{(3)}$	$C_{n-b-1}^{(4)}$	
sum over	$b \in [5, n-1]$	$b \in [4, n-1]$	
case total	$\Delta(7C_{n-6}^{(13)} + 6C_{n-8}^{(16)}$ $+C_{n-10}^{(19)})$	$C_{n-6}^{(12)} + 2C_{n-8}^{(15)} + C_{n-10}^{(18)}$ $+2C_{n-7}^{(14)} + 2C_{n-5}^{(10)} - C_{n-5}^{(8)}$	$C_{n-4}^{(6)}$

Case	4	5	6
pattern type	$b_1 < b_2 = b_3 = b_4 \equiv b$ $c_2 = c_3 = c_4 \equiv c$	common b's common a's	common b's $c_1 = c_2 < c_3 = c_4$
position of b	$b-2$	$b+3$	b
$\rho =$	reduce$(W_1 a_2 a_3 a_4)$	reduce$(W_1 a)$	reduce$(W_1 a_2 a_3)$
ranges over	$P_{321}^{(1)}(b)$	$P_{321}^{(0)}(b+3)$	$P_{321}^{(0)}(b+1)$
subject to	last 3 $\uparrow$ and max occurs earlier	last $\leq$ max-4	two largest 3 or more from end
a set of size	$C_{b-5}^{(7)} + C_{b-7}^{(10)}$	$C_{b-2}^{(6)}$	$C_{b-3}^{(5)}$
$\sigma =$	reduce(cW_2)	reduce$(c_1 c_2 c_3 c_4 W_2)$	reduce$(c_3 c_4 W_2)$
ranges over	$P_{321}^{(0)}(n-b+3)$	$P_{321}^{(0)}(n-b+1)$	$P_{321}^{(0)}(n-b+2)$
subject to	$\sigma_1 \geq 4$	1 weakly after pos 5	1 and 2 weakly after pos 3
a set of size	$C_{n-b-1}^{(5)}$	$C_{n-b-4}^{(6)}$	$C_{n-b-2}^{(5)}$
sum over	$b \in [5, n-1]$	$b \in [2, n-4]$	$b \in [3, n-2]$
case total	$C_{n-6}^{(12)} + C_{n-8}^{(15)}$	$C_{n-6}^{(12)}$	$C_{n-5}^{(10)}$

Grand Total: $13C_{n-6}^{(12)} + 19C_{n-8}^{(15)} + 9C_{n-10}^{(18)} + C_{n-12}^{(21)} + 4C_{n-7}^{(14)} + 5C_{n-5}^{(10)} + C_{n-4}^{(6)} - 2C_{n-5}^{(8)}$

8 132 Patterns in 123-Avoiding Permutations

Since we are henceforth dealing with 123-avoiding permutations, let $P^{(i)}(n)$ denote the set of 123-avoiding permutations on $[n]$ that contain exactly i 132 patterns. Here we compute $|P^{(i)}(n)|$ for $1 \leq i \leq 4$. We distinguish cases according to the rightmost c among the i 132 patterns. Say this c does single duty if it appears as the "c" of only one 132 pattern, double duty if it is the "c" of two 132 patterns, and so on. To start the ball rolling, we have the following count for $P^{(0)}(n)$. A proof is included for completeness.

Theorem 30 ([9]). *The number of permutations on* $[n]$ *that avoid* 123 *and* 132 *is* 2^{n-1}.

Proof. For π on $[n]$ to avoid 123 and 132 patterns, π_1 must be n or $n-1$ for if it were $\leq n-2$ it would have two larger successors, forcing π_1 to initiate a proscribed pattern. Similarly, $\pi_k \geq n-k$ for all $1 \leq k \leq n-1$ and, conversely, this condition guarantees that π avoids 123 and 132. The number of such permutations is the permanent of the lower Hessenberg matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} n & n-1 & n-2 & \ldots & \ldots & 1 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix} \end{array}$$

which is well known to be 2^{n-1}. □

We now proceed to count $P^{(i)}(n)$ for $1 \leq i \leq 4$, considering separate cases according to pattern overlap and summarizing computations in tables similar to those in the last section. The reader is left to verify details.

No 123, One 132 Pattern

By Lemma 16, π has the form $W_1\, a\, c\, W_2$ with $b \in W_2$ and $b = a+1$. Also, W_1 comprises the letters $[b+1, n]\backslash\{c\}$ and W_1 comprises $[1, b]\backslash\{a\}$. For $b \in [2, n-1]$, the map $\pi \to (\rho, \sigma)$ is a bijection where $\rho = \text{reduce}(W_1 c)$ ranges over $P^{(0)}(n-b)$, a set of

size 2^{n-b-1}, and $\sigma = \text{reduce}(W_2)$ ranges over $P^{(0)}(b-1)$, a set of size 2^{b-2}. Summing over $b \in [2, n-1]$, we have

Theorem 31 (Robertson [10]). $|P^{(1)}(n)| = \binom{n-2}{1} 2^{n-3}$.

No 123, Two 132 Patterns

Here the two cases are (1) c_2 does single duty, that is, $c_1 \neq c_2$, (2) c_2 does double duty, that is, $c_1 = c_2$. In case (1) π has the form $W_1\, a_2\, c_2\, W_2$ with $b_1 \in W_1 \cup \{c_2\}$, $a_1, c_1 \in W_1$, $b_2 \in W_2$ and with a_2, b_2 consecutive in value and $b_2 < b_1$ (else $a_1 b_1 b_2$ is a 123). Case (2) forces common b's as well as common c's, and π has the form $W_1\, a_1\, a_2\, c\, W_2$ with $b = b_1 = b_2 \in W_2$ and a_2, a_1, b consecutive in value.

$\{\pi \in P^{(2)}(n)\}$

Case	1	2
pattern type	$c_1 \neq c_2$	$c_1 = c_2$
π has form	$W_1\, a_2\, c_2\, W_2$	$W_1\, a_1\, a_2\, c\, W_2$
$a, c, b =$	a_2, c_2, b_2	obvious
W_1 comprises	$[b+1, n] \backslash \{c\}$	$[b+1, n] \backslash \{c\}$
W_2 comprises	$[1, b] \backslash \{a\}$	$[1, b] \backslash \{a_1, a_2\}$
$\rho =$	$\text{reduce}(W_1\, c)$	$\text{reduce}(W_1\, c)$
ranges over	$P^{(1)}(n-b)$	$P^{(0)}(n-b)$
a set of size	$\binom{n-b-2}{1} 2^{n-b-3}$	2^{n-b-1}
$\sigma =$	$\text{reduce}(W_2)$	$\text{reduce}(W_2)$
ranges over	$P^{(0)}(b-1)$	$P^{(0)}(b-2)$
a set of size	2^{b-2}	2^{b-3}
sum over	$b \in [2, n-3]$	$b \in [3, n-1]$
case total	$\binom{n-3}{2} 2^{n-5}$	$\binom{n-3}{1} 2^{n-4}$

Grand Total: $\binom{n-3}{1} 2^{n-4} + \binom{n-3}{2} 2^{n-5}$

No 123, Three 132 Patterns

Here there are four cases: (1) c_3 does single duty and b_3 is the only pattern letter following c_3, (2) c_3 does single duty and $b_2 = b_3$, (3) $c = c_2 = c_3$ does double duty, (4) $c = c_1 = c_2 = c_3$ does triple duty.

In case (1) $\pi = W_1\, a\, c\, W_2$ with $a = a_3,\ c = c_3,\ b = b_3 \in W_2$ and a, b consecutive in value.

In case (2) only four distinct letters are involved among the three 132 patterns (their reduced form is 1432 with patterns $1\,4\,3, 1\,4\,2, 1\,3\,2$) and $\pi = W_1\, a\, c\, b\, W_2$ with $a = a_1 = a_2 = a_3,\ c = c_1 = c_2,\ b = b_1 = c_3$ and $b_2 = b_3 \in W_2$. Also a, b_2, b consecutive in value.

In case (3) $\pi = W_1\, a_2\, a_3\, c\, W_2$ with $b = b_2 = b_3 \in W_2$ and a_3, a_2, b consecutive in value.

In case (4) $\pi = W_1\, a_1\, a_2\, a_3\, c\, W_2$ with $b = b_1 = b_2 = b_3 \in W_2$ and all four of a_3, a_2, a_1, b consecutive in value.

$\boxed{\{\pi \in P^{(3)}(n)\}}$

Case	1	2
pattern type	c_3 single duty, $b_2 \neq b_3$	c_3 single duty, $b_2 = b_3$
π has form	$W_1\, a\, c\, W_2$	$W_1\, a\, c\, b\, W_2$
$a, c, b =$	a_3, c_3, b_3	a_1, c_1, b_1
W_1 comprises	$[b+1, n]\backslash\{c\}$	$[b+1, n]\backslash\{c\}$
W_2 comprises	$[1, b]\backslash\{a\}$	$[1, b-1]\backslash\{a = b-2\}$
$\rho =$	reduce$(W_1\, c)$	reduce$(W_1\, c)$
ranges over	$P^{(2)}(n-b)$	$P^{(0)}(n-b)$
a set of size	$\binom{n-b-3}{1}2^{n-b-4} + \binom{n-b-3}{2}2^{n-b-5}$	2^{n-b-1}
$\sigma =$	reduce(W_2)	reduce(W_2)
ranges over	$P^{(0)}(b-1)$	$P^{(0)}(b-2)$
a set of size	2^{b-2}	2^{b-3}
sum over	$b \in [2, n-4]$	$b \in [3, n-1]$
case total	$\binom{n-4}{2}2^{n-6} + \binom{n-4}{3}2^{n-7}$	$\binom{n-3}{1}2^{n-4}$

$\{\pi \in P^{(3)}(n)\}$ continued

Case	3	4
pattern type	c_3 double duty	c_3 triple duty
π has form	$W_1\, a_2\, a_3\, c\, W_2$	$W_1\, a_1\, a_2\, a_3\, c\, W_2$
$a, c, b =$	a_3, c_3, b_3	obvious
W_1 comprises	$[b+1, n]\backslash\{c\}$	$[b+1, n]\backslash\{c\}$
W_2 comprises	$[1, b]\backslash\{a_2, a_3\}$	$[1, b]\backslash\{a_1, a_2, a_3\}$
$\rho =$	reduce$(W_1\, c)$	reduce$(W_1\, c)$
ranges over	$P^{(1)}(n-b)$	$P^{(0)}(n-b)$
a set of size	$\binom{n-b-2}{1}2^{n-b-3}$	2^{n-b-1}
$\sigma =$	reduce(W_2)	reduce(W_2)
ranges over	$P^{(0)}(b-2)$	$P^{(0)}(b-3)$
a set of size	2^{b-3}	2^{b-4}
sum over	$b \in [3, n-3]$	$b \in [4, n-1]$
case total	$\binom{n-4}{2}2^{n-6}$	$\binom{n-4}{1}2^{n-5}$

Grand Total: $\binom{n-3}{1}2^{n-4} + \binom{n-3}{2}2^{n-5} + \binom{n-4}{3}2^{n-7}$

No 123, Four 132 Patterns

The 7 cases to be considered here are

Case 1. c_4 single duty, b_4 is the only pattern letter occurring after c_4. Here $\pi = W_1\, a\, c\, W_2$ with $a = a_4$, $c = c_4$, $b = b_4 \in W_2$ and $b = a + 1$.

Case 2. c_4 single duty, $b_3 = b_4$, only two distinct c's. Here there are only 5 distinct pattern letters and the pattern overlap is as in the figure below with a, c, b consecutive in position (all entries in the same column coincide).

	a	c	b	
a_1		c_1	b_1	
	a_2	c_2	b_2	
	a_3	c_3		b_3
	a_4		c_4	b_4

Writing $\pi = W_1\, a\, c\, b\, W_2$ we must also have all four of a, b_3, a_1, b consecutive in value: $a = b-3,\ b_3 = b-2,\ a_1 = b-1$.

Case 3. c_4 single duty, $b_3 = b_4$, three distinct c's (four distinct c's is not possible). Here there are 6 distinct pattern letters and the pattern overlap is as in the figure below again with a, c, b consecutive in position.

		a	c	b	
a_1	c_1		b_1		
		a_2	c_2	b_2	
		a_3	c_3		b_3
		a_4		c_4	b_4

Writing $\pi = W_1\, a\, c\, b\, W_2$, we have $b_3\,(= b_4)$, b consecutive in value: $b_3 = b-1$.

Case 4. c_4 double duty, only 5 distinct pattern letters. The pattern overlap now has the first four pattern letters consecutive in position.

$b-1$	c_2	$b-2$	c_4	b
a_1	c_1		b_1	
a_2	c_2			b_2
a_3			c_3	b_3
		a_4	c_4	b_4

Case 5. c_4 double duty, more than 5 distinct pattern letters. Here $c = c_3 = c_4,\ b = b_3 = b_4$ and $\pi = W_1\, a_3\, a_4\, c\, W_2$ with a_4, a_3, b consecutive in value: $a_4 = b-2$, $a_3 = b-1$.

Case 6. $c = c_2 = c_3 = c_4$ triple duty. Here $\pi = W_1\, a_2\, a_3\, a_4\, c\, W_2$ with $b = b_2 = b_3 = b_4 \in W_2$ and a_4, a_3, a_2, b consecutive in value.

Case 7. $c = c_1 = c_2 = c_3 = c_4$ quad duty. Here $\pi = W_1\, a_1\, a_2\, a_3\, a_4\, c\, W_2$ with $b = b_1 = b_2 = b_3 = b_4 \in W_2$ and a_4, a_3, a_2, a_1, b consecutive in value.

The computation tables follow.

$\{\pi \in P^{(4)}(n)\}$

Case	1	2	3
pattern type	c_4 single duty, $b_3 \neq b_4$	c_4 single duty, $b_3 = b_4$, only 2 distinct c's	c_4 single duty, $b_3 = b_4$, 3 distinct c's
π has form	$W_1\, a\, c\, W_2$	$W_1\, a\, c\, b\, W_2$	$W_1\, a\, c\, b\, W_2$
W_1 comprises	$[b+1, n]\backslash\{c\}$	$[b-1, n]\backslash\{b, c\}$	$[b+1, n]\backslash\{c\}$
W_2 comprises	$[1, b]\backslash\{a\}$	$[1, b-2]\backslash\{a = b-3\}$	$[1, b-1]\backslash\{a\}$
$\rho =$	reduce$(W_1\, c)$	reduce$(W_1\, c)$	reduce$(W_1\, c)$
ranges over	$P^{(3)}(n-b)$	$\{\rho \in P^{(0)}(n-b+1) :$ last $\neq$ min$\}$	$P^{(1)}(n-b)$
a set of size	$\binom{n-b-3}{1}2^{n-b-4} +$ $\binom{n-b-3}{2}2^{n-b-5} +$ $\binom{n-b-4}{3}2^{n-b-7}$	2^{n-b-1}	$\binom{n-b-2}{1}2^{n-b-3}$
$\sigma =$	reduce(W_2)	reduce(W_2)	reduce(W_2)
ranges over	$P^{(0)}(b-1)$	$P^{(0)}(b-3)$	$P^{(0)}(b-2)$
a set of size	2^{b-2}	2^{b-4}	2^{b-3}
sum over	$b \in [2, n-4]$	$b \in [4, n-1]$	$b \in [3, n-3]$
case total	$\binom{n-4}{2}2^{n-6} +$ $\binom{n-4}{3}2^{n-7} +$ $\binom{n-5}{4}2^{n-9}$	$\binom{n-4}{1}2^{n-5}$	$\binom{n-4}{2}2^{n-6}$

$\{\pi \in P^{(4)}(n)\}$ continued

Case	4	5
pattern type	c_4 double duty only 5 distinct pattern letters	c_4 double duty > 5 distinct pattern letters
π has form	$W_1\, b{-}1\, c_2\, b{-}2\, c_4\, W_2$	$W_1\, b{-}1\, b{-}2\, c\, W_2$
W_1 comprises	$[b+1, n]\backslash\{c_2, c_4\}$	$[b+1, n]\backslash\{c\}$
W_2 comprises	$[1, b-3] \cup \{b\}$	$[1, b-3] \cup \{b\}$
$\rho =$	reduce$(W_1\, c_2\, c_4)$	reduce$(W_1\, c)$
ranges over	$\{\rho \in P^{(0)}(n-b)$: last 2 $\downarrow\}$	$P^{(2)}(n-b)$
a set of size	2^{n-b-2}	$\binom{n-b-3}{1}2^{n-b-4} + \binom{n-b-3}{2}2^{n-b-5}$
$\sigma =$	reduce(W_2)	reduce(W_2)
ranges over	$P^{(0)}(b-2)$	$P^{(0)}(b-2)$
a set of size	2^{b-3}	2^{b-3}
sum over	$b \in [3, n-2]$	$b \in [3, n-4]$
case total	$\binom{n-4}{1}2^{n-5}$	$\binom{n-5}{2}2^{n-7} + \binom{n-5}{3}2^{n-8}$

Case	6	7
pattern type	c_4 triple duty	c_4 quadruple duty
π has form	$W_1\, a_2\, a_3\, a_4\, c\, W_2$	$W_1\, a_1\, a_2\, a_3\, a_4\, c\, W_2$
W_1 comprises	$[b+1, n]\backslash\{c\}$	$[b+1, n]\backslash\{c\}$
W_2 comprises	$[1, b]\backslash\{a_2, a_3, a_4\}$	$[1, b]\backslash\{a_1, a_2, a_3, a_4\}$
$\rho =$	reduce$(W_1\, c)$	reduce$(W_1\, c)$
ranges over	$P^{(1)}(n-b)$	$P^{(0)}(n-b)$
a set of size	$\binom{n-b-2}{1}2^{n-b-3}$	2^{n-b-1}
$\sigma =$	reduce(W_2)	reduce(W_2)
ranges over	$P^{(0)}(b-3)$	$P^{(0)}(b-4)$
a set of size	2^{b-4}	2^{b-5}
sum over	$b \in [4, n-3]$	$b \in [5, n-1]$
case total	$\binom{n-5}{2}2^{n-7}$	$\binom{n-5}{1}2^{n-6}$

Grand Total: $2\binom{n-4}{1}2^{n-5} + 3\binom{n-4}{2}2^{n-6} + \binom{n-4}{3}2^{n-7} + \binom{n-5}{3}2^{n-8} + \binom{n-5}{4}2^{n-9}$

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