A Combinatorial Proof of Sun's "Curious" Identity

David Callan
Department of Statistics, University of Wisconsin-Madison
1210 W. Dayton Street, Madison, WI 53706-1693
callan@stat.wisc.edu

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Abstract

A binomial coefficient identity due to Zhi-Wei Sun is the subject of half a dozen recent papers that prove it by various analytic techniques and establish a generalization. Here we give a simple proof that uses weight-reversing involutions on suitable configurations involving dominos and colorings. With somewhat more work, the method extends to the generalization also.

0 Introduction

The identity

$$\sum_{i=0}^{m}(x+m+1)(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}-\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=(x-m)\binom{x}{m} \tag{1}$$

was proved [1, 2, 3, 4, 5] using induction, generating functions, Riordan arrays, the WZ method, and the so-called Jensen formula respectively. This identity is the case $z=1$ of the following generalization [6]:

$$\begin{aligned}&(x+(m+1)z)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n} \\ &=z\sum_{0\le l\le n\le m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}+(x-m)\binom{x}{m}.\end{aligned} \tag{2}$$

Section 1 is a "lite" version of the paper: a proof of (1) using weight-reversing involutions on configurations on an initial segment of the positive integers that involve some vertices covered by dominos and the rest colored black or white. Section 2 then uses the same method in a somewhat more complicated setting to prove (2).

1 Identity (1)

Both sides of (1) are polynomials in x of degree $m+1$ with the same leading coefficient, $1/m!$, and so it suffices to prove (1) for $m+1$ distinct values of x. The right hand side vanishes for $x = 0, 1, 2, \ldots, m$ and so, putting $x = m-k$, replacing the summation index i by $m-i$ and canceling a $(-1)^m$ factor, (1) is equivalent to

$$(2m-k+1)\sum_{i=0}^{m}(-1)^i\binom{2m+y-k-i}{i}\binom{2m+y-2i}{m-i} = \sum_{i=0}^{m}(-1)^i\binom{2m-k-i}{i}2^{2(m-i)} \qquad 0 \le k \le m.$$

We show that in fact both sides $= (2m-k+1)2^k$.

After some cancellation, this amounts to

$$\sum_{i=0}^{m}(-1)^i\binom{2m+y-k-i}{i}\binom{2m+y-2i}{m-i} = 2^k, \tag{3}$$

and

$$\sum_{i=0}^{m}(-1)^i\binom{2m-k-i}{i}2^{2m-2i} = (2m-k+1)2^k \tag{4}$$

Fix an integer $k \in [0, m]$, set $y = b$, a fixed nonnegative integer and let $[n]$ denote $\{1, 2, \ldots, n\}$. The right hand side of (3) is the total weight of what we will call Sun(3)-configurations on $[2m+b]$ defined as follows: cover some of $[2m+b-k]$ with i $(0 \le i \le m)$ nonoverlapping dominos, each covering two adjacent vertices $[\binom{2m+b-k-i}{i}$ choices] and then color $m-i$ of the remaining $2m+b-2i$ vertices black and the rest white $[\binom{2m+b-2i}{m-i}$ choices]. Assign weight $= (-1)^{\#\text{dominos}}$. Two examples are given in the Figure below (both for $m=4$, $b=2$, $k=1$ and so $2m+b-k=9$; an underline denotes a black vertex). The first has weight -1, the second has weight $+1$.

$\underline{1}\ \underline{2}\ \underline{3}\ 4\ 5\ 6\ \boxed{7}\boxed{8}\ 9\ \underline{10}$ (domino range)

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The left hand side of (3) is the total weight of all Sun(3)-configurations. Here is a weight-reversing involution on most of them. Call $[2m+b-k]$ the *domino range* and say two consecutive vertices form an *active pair* if both lie in the domino range and are either (i) covered by a domino or (ii) colored black, white in that order. Look for the leftmost active pair of vertices. If active by virtue of (i), remove the domino and color the left vertex black, the right one white. If active by virtue of (ii), remove the colors and cover them with a domino. For example, this map interchanges the two configurations above. The map preserves # dominos + # black vertices while altering # dominos by 1 and hence is a weight-reversing involution on all Sun(3)-configurations except those with no active pairs. Such an exceptional configuration entails no dominos ($i=0$) and hence weight = 1, and all black vertices in the domino range flush left. When $i=0$, the vertices outside the domino range may be colored arbitrarily ($j \in [0,k]$ black vertices outside implies $m-k+b+j$ white vertices inside and $m-k+b+j \geq 0$ since $k \leq m$) and the flush left requirement then determines the configuration. Hence 2^k exceptional configurations, and (3) follows.

Identity (4) can be proved similarly using the following configurations: place $i \geq 0$ dominos on $[2m-k]$ [$\binom{2m-k-i}{i}$ choices] but now color the remaining $2m-2i$ vertices black or white independently [2^{2m-2i} choices]. Assign weight = $(-1)^{\#\text{dominos}}$ and use the same map. Again, the exceptional configurations are those with no dominos and all black vertices in the domino range flush left. The number of such configurations is now $(2m-k+1)$ [locate switchover from black to white in domino range] $\times 2^k$ [color vertices outside domino range], and (4) follows.

2 Identity (2)

Just as for (1) and with the same change of variable, (2) is equivalent to

$$\begin{aligned}((m-k)+(m+1)z)\sum_{i=0}^{m}(-1)^{(m-i)}\binom{m-k+y+(m-i)z}{i}\binom{y+(m-i)(z+1)}{m-i}\\ = \; z\sum_{0\leq j\leq i\leq m}(-1)^i\binom{i}{j}\binom{m-k+j}{m-i}(1+z)^{i+j}(1-z)^{i-j} \qquad 0\leq k\leq m\end{aligned}$$

Again, both sums have a closed form and it suffices to show that with $y=b$ a

nonnegative integer and $z = q$ also a nonnegative integer,

$$\sum_{i=0}^{m}(-1)^i\binom{(q+1)m-k+b-qi}{i}\binom{(q+1)m+b-(q+1)i}{m-i} \;=\; q^{m-k}(1+q)^k \quad (5)$$

and

$$\sum_{0\le j\le i\le m}(-1)^{m-i}\binom{i}{j}\binom{m-k+j}{m-i}(1+q)^{i+j-k}(1-q)^{i-j}$$
$$= \big((m-k)+(m+1)q\big)q^{m-k-1} \quad (6)$$

both for $0 \le k \le m$.

For (5), the analog of a Sun(3)-configuration uses $(q+1)$-ominos i.e., $q+1$ consecutive vertices: choose $i \le m$ $(q+1)$-ominos in the "omino range" $[(q+1)m+b-k]$, choose $m-i$ of the remaining $(q+1)m+b-(q+1)i$ vertices in $[(q+1)m+b]$ to color black (B) and the rest white (W). Use weight $= (-1)^{\#\text{ominos}}$. Here the active $(q+1)$-tuples in the omino range are $(q+1)$-ominos and strings W^qB $(= \underbrace{W\ldots W}_{q}B)$. The analogous involution is obvious and the survivors are again the configurations with no active $(q+1)$-tuple. This entails no omino (so $i=0$ and weight $=1$) and no W^qB in $[(q+1)m+b-k]$. Such a configuration has arbitrary coloring of the last k vertices, say j Bs with $0 \le j \le k$ and then begins $W^{i_1}BW^{i_2}B\ldots W^{i_{m-j}}BW\ldots W$ with each exponent $i_\ell \in [0, q-1]$. We find that there are $\sum_{j=0}^{k}\binom{k}{j}$ [choose Bs among last k vertices] $\times q^{m-j}$ [choose exponents]$= q^{m-k}(1+q)^k$ survivors, as expected.

The form of the right hand side of (6) suggests separate treatment of the case $k=m$:

$$\sum_{0\le j\le i\le m}(-1)^{m-i}\binom{i}{j}\binom{j}{m-i}(1+q)^{i+j-m}(1-q)^{i-j} = m+1$$

Here, the relevant configurations are 5-tuples (i, J, K, A, B) with $i \in [0,m]$, J a j-subset of $I := [i]$, K an $(m-i)$-subset of J, A an arbitrary q-colored subset of $J\backslash K$ and B an arbitrary q-colored subset of $I\backslash J$. The weight is $(-1)^{m-i+|B|}$. Since the value of i is built into K, whose size is $m-i$, and the conditions $J, B \subseteq I$ say that $\max\{J,B\} \le m-|K|$, these configurations may be more compactly described as 4-tuples (J, K, A, B) satisfying $J \subseteq [m]$, both K and $A \subseteq J$, $K \cap A = \emptyset$, $J \cap B = \emptyset$, A

and B both q-colored, and finally, $\max\{J, B\} \leq m - |K|$. The weight is $(-1)^{|K|+|B|}$. Such a 4-tuple can be conveniently represented by a $2 \times m$ 0-1 matrix with (possibly) some underlined 0s: 1s in the top (resp. bottom) row indicate elements of K (resp. J) and underlined 0s in the top (resp. bottom) row indicate elements of A (resp. B). The weight is then $(-1)^{\#\text{ 1s in top row} + \#\ \underline{0}\text{s in bottom row}}$. Of course this does not specify any element's color but no matter: that color will never be changed. For example, with $m = 10$,

$$\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{pmatrix} 0 & \underline{0} & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & \underline{0} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

represents $K = \{3, 5, 6\},\ J = \{1, 2, 3, 5, 6\},\ A = \{2\},\ B = \{7\}$. Each column is one of $\binom{0}{0}, \binom{0}{1}, \binom{1}{1}, \binom{\underline{0}}{1}, \binom{0}{\underline{0}}$ and the only further restriction is that there must be at least as many plain 0s terminating the bottom row as 1s in the top row.

The weights of all configurations containing a colored vertex (= underlined matrix entry) cancel out: Look for the leftmost underline and if its column is $\binom{\underline{0}}{\underline{0}}$ then change that column to $\binom{\underline{0}}{1}$ and vice versa. Now for a weight-reversing involution on matrix configurations with no underlines. The survivors are those with $K = \emptyset$ (i.e. top row all 0s) and J a (possibly empty) initial segment of $[m]$ (i.e. bottom row is 1s followed by 0s). There are $m + 1$ such, all of weight 1. The involution on the others is as follows. Look for the first $\binom{1}{1}$ column and consider v, the list of entries in the bottom row following this $\binom{1}{1}$; v = all of bottom row if there is no $\binom{1}{1}$ column and note that, in this case, v must contain a 01 string since J is not an initial segment of $[m]$. If v has no 01 string (and hence v has the form 1^i0^j with $i \geq 0$), replace the $\binom{1}{1}$ by $\binom{0}{0}$ and change the first 0 in v to 1. There must be such a 0 since $\max\{J\} \leq m - |K|$. On the other hand, if v has a 01 string, change the column of the 0 in v's last 01 string to $\binom{1}{1}$ and change the last 1 in v to 0. An example is illustrated.

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \longleftrightarrow \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad (7)$$

$v = 0\,1\,0\,1\,1\,1\,0$ contains a 01 string
The 0 of the last 01 in v is in column 5
v's last 1 is in column 8

$v = 1\,1\,0\,0$ contains no 01 string
The last $\binom{1}{1}$ is in column 5
v's first 0 is in column 8

Finally, we do the case $0 \le k < m$:

$$\sum_{0\le j\le i\le m} (-1)^{m-i}\binom{i}{j}\binom{m-k+j}{m-i}(1+q)^{i+j-k}(1-q)^{i-j}$$
$$= \big((m-k)+(m+1)q\big)q^{m-k-1}$$

Here we need to consider the $(m-k)$-set $E = [m+1, 2m-k]$ as well as $M = [m]$. The 5-tuple configurations are (i, J, K, A, B) with $i \in [0, m]$, J a j-subset of $I := [i]$, K an $(m-i)$-subset of $J \cup E$, A an arbitrary q-colored subset of $(J \cup E)\backslash K$ and B an arbitrary q-colored subset of $I\backslash J$, and weight $= (-1)^{|K|+|B|}$. Again we can drop the "i" by imposing the condition $\max\{J, B\} \le m - |K|$ and we will work with the equivalent matrix formulation which is the same as before except now the first row has an extra $m-k$ entries. Specifically, we have a $2 \times m$ main matrix each column being $\binom{0}{0}$, $\binom{0}{1}$, $\binom{1}{1}$, $\binom{\underline{0}}{1}$, or $\binom{0}{\underline{0}}$ as before, an $(m-k)$-vector (considered as an extension of the top row of the matrix) each entry being $0, 1$ or $\underline{0}$, and the restriction that there are at least as many plain 0s terminating the bottom row as there are 1s in the top row and vector combined. We have to expect complications due to the relatively complicated right hand side, $\big((m-k)+(m+1)q\big)q^{m-k-1}$, and what we'll do is kill off weights in four steps, pruning our class of configurations at each step except for some easily counted survivors, each of weight 1.

First, our earlier $\binom{0}{\underline{0}} \leftrightarrow \binom{\underline{0}}{1}$ involution kills off underlines in the main matrix. Henceforth only the $(m-k)$-vector, call it v, can contain underlines. Second, the bijection of (7) kills all configurations where either the main matrix contains either $\binom{1}{1}$ or its bottom row contains 01 (or both).

We have now pruned our configurations to those with a 0-1 (no underlines) matrix with top row all 0s, and all 1s (if any) in the bottom row flush left. Those for which v is all 0s are survivors—$(m+1)q^{m-k}$ of them. Otherwise, v has the form $\underline{0}^i a w$ with $i \ge 0$, $a = 0$ or 1, and w a (possibly empty) vector of 1s, 0s, $\underline{0}$s. Consider the configurations with bottom row all 0s and $a = 0$. If w in $\underline{0}^i a w$ is all $\underline{0}$s, we have a survivor—$(m-k)q^{m-k-1}$ of them. Otherwise, flipping the first 0/1 entry in w ($0 \leftrightarrow 1$) is our involution. All that's left are configurations with (i) bottom row of the form $1^i 0^{m-i}$ with $i \ge 1$ or (ii) $a = 1$ (or both). Here the involution is: if $a = 1$, change it to 0 and change the bottom row to $1^{i+1}0^{m-(i+1)}$ (there will be room for this extra 1

in the bottom row since $a = 1$ forces a trailing 0 in the bottom row), and if $a = 0$, change it to 1 and change the bottom row to $1^{i-1}0^{m-(i-1)}$. We are done.

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