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The Generation of $Sp(\mathbb{F}_2)$ by Transvections

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In 1955, in [2], Dieudonné, observing that each symplectic group $Sp(V)$ is generated by its transvections, considered the problem of finding the minimal number of factors in the expression of elements σ of $Sp(V)$ as products of transvections in $Sp(V)$.[1] The answer is in general residue σ, but with some classes of exceptions. Complications arise when the underlying field is $\mathbb{F}_2$, and Dieudonné's list of exceptions [2, p. 164] for this case is incomplete. The difficulty occurs in 4.4° page 163 and seems to lie in the fact that, following the notation there, the plane T chosen orthogonal to c and d need not (in fact, cannot) be orthogonal to the revised "c" and "d," as is tacitly assumed. The proof in 4.4° fails and there is a class of exceptions of residue 5 in dimension 6 (see 3.3 below).

This paper is devoted to completing the list of exceptions for $\mathbb{F}_2$. Also mentioned are the minor corrections thus necessitated in the theory for the orthogonal group in characteristic 2, treated in [2, p. 177].

Section 1 is introductory, establishing notation and the basic facts needed. Section 2 recasts results of [2]. Section 3 presents the "new" exceptions and Section 4 shows there are no more. Section 5 collects results to state the main theorem. Section 6 considers the orthogonal case.

1. PRELIMINARIES

Our notation generally follows [4, 5]. V denotes a nonzero left vector space over the commutative field F, equipped with a regular alternating sesquilinear form $(\ ,\ )$. So $\dim_F V$ is even, $2m$ say, and the group of isometries of V is the symplectic group $Sp_{2m}(V)$. $\mathfrak{X}$, always consisting of $x_1, \ldots, x_m$,

[1] The analogous question for not necessarily regular symplectic spaces has recently been treated for the case "the characteristic of the underlying field is not equal to 2," by Spengler and Wolff [7].

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$y_1, \ldots, y_m$, denotes a symplectic base for V. A *symplectic transvection* is a transvection in $\mathrm{Sp}_{2m}(V)$. σ is an element of a symplectic group, ω denotes $\sigma - 1$. $P_\sigma = \ker \omega$ is the *fixed space* of σ, $R_\sigma = \dim \omega$ is the *residual space* of σ, $\operatorname{res} \sigma = \dim R_\sigma$ is the *residue* of σ. The subscripts are often omitted when the σ is clear from the context. An element σ is *exceptional* iff it cannot be expressed as a product of $\operatorname{res} \sigma$ symplectic transvections. Clearly exceptional elements occur in conjugacy classes. For $\sigma \in \mathrm{Sp}(V)$, the *dimension* of σ is $\dim \sigma = \dim_F V$. If $P_\sigma \subseteq R_\sigma$, the *level* of σ is $\operatorname{res} \sigma - \frac{1}{2} \dim \sigma$; thus $0 \leqslant \text{level } \sigma \leqslant \frac{1}{2} \dim \sigma$ and level is defined only when $P_\sigma \subseteq R_\sigma$. Neither component may vanish in an orthogonal sum of symplectic elements $\sigma_1 \perp \sigma_2$, unless so stated. We say $\sigma \in \mathrm{Sp}(V)$ is *indecomposable* iff σ cannot be written in the form $\sigma_1 \perp \sigma_2$. 1_U is the identity map on U. Suppose $\{A_i\}_{i=1}^k$, $k \geqslant 1$, are square matrices over F. We say $\sigma \in \mathrm{Sp}(V)$ is *of class* $A_1 \perp A_2 \perp \cdots \perp A_k$ iff V can be written as an orthogonal sum of subspaces $\{U_i\}_{i=1}^k$ preserved by σ such that there exists for each i, a symplectic base $\mathfrak{X}_i$ for U_i with $m_{\mathfrak{X}_i}(\sigma \mid_{U_i}) = A_i$. It is easy to see the set of all σ of class $A_1 \perp A_2 \perp \cdots \perp A_k$ is either void or a full conjugacy class in $\mathrm{Sp}(V)$.

We first note the useful identities: $(\sigma x, \omega y) = -(\omega x, y)$ and $(\omega x, \omega y) = -[(x, \omega y) + (\omega x, y)]$, and the following fact. If U is a subspace of V preserved by σ, then $\omega(x) \in U^*$ implies $x \in [\omega(U)]^*$. (Note the foregoing holds for arbitrary sesquilinear forms.) The following two facts reduce the problem to the determination of the exceptional classes.

1.1. *An element $\sigma \in \mathrm{Sp}(V)$ cannot be expressed as a product of fewer than* $\operatorname{res} \sigma$ *transvections* [4] *and can always be expressed as a product of not more than* $\operatorname{res} \sigma + 1$ *symplectic transvections* [5].

We can also assume $P_\sigma \subseteq R_\sigma$ as follows. Take T supplementary to $\operatorname{rad} P_\sigma$ in P_σ. Then T is regular. $\sigma^* = \sigma \mid_{T^*}$ has fixed space $\operatorname{rad} P_\sigma$, residual space R_σ and thus $P_{\sigma^*} \subseteq R_{\sigma^*}$. Now $\sigma = 1_T \perp \sigma^*$ and it is easily seen σ is exceptional iff σ^* is. Hence

1.2. *To find all exceptions, it is only necessary to find those with $P \subseteq R$.*

The following is the key fact for the investigation.

1.3. *Let $\sigma \in \mathrm{Sp}_{2m}(V)$, τ a transvection in $\mathrm{Sp}_{2m}(V)$. Then* $\operatorname{res} \tau\sigma < \operatorname{res} \sigma$ *iff τ can be written as $\tau_{\omega a, (\omega a, a)^{-1}}$ for some $a \in V$ with $(\omega a, a) \neq 0$* [5]. *Hence σ is exceptional iff $\tau_{\omega a, (\omega a, a)^{-1}}\sigma$ is exceptional for all $a \in V$ such that $(\omega a, a) \neq 0$.*

We say $a \in V$ is *admissible* for σ iff $(\omega a, a) \neq 0$. What if σ has no admissible vectors? (Clearly σ is then exceptional and in fact these comprise all exceptions when $F \neq \mathbb{F}_2$ [5].) In this case σ is an involution, $\sigma_{|R} = -1$ and (i) if $\chi(F) \neq 2$, then σ is of the form $1_P \perp -1_R$, (ii) if $\chi(F) = 2$, then $R \subseteq P$. Thus if in addition $P \subseteq R$, (i) if $\chi(F) \neq 2$, then $\sigma = -1_V$ and level $\sigma = m$, (ii) if

$\chi(F) = 2$, then $P = R$ and level $\sigma = 0$. We say σ is a *hyperbolic involution* iff σ has no admissible vectors in V. Suppose now a is an admissible vector for σ and consider $\sigma_1 = \tau_{\omega a,(\omega a,a)-1}\sigma$. We have (i) if $a \in R_\sigma$, then $P_{\sigma_1} \subseteq R_{\sigma_1}$, $\dim \sigma_1 = \dim \sigma$ and level $\sigma_1 =$ level $\sigma - 1$, (ii) if $a \notin R_\sigma$, then $\sigma_1 = 1_T \perp \sigma_1{}^*$ where T is a regular plane and $\sigma_1{}^* \in \mathrm{Sp}_{2m-2}(T^*)$, $P_{\sigma_1{}^*} \subseteq R_{\sigma_1{}^*}$, $\dim \sigma_1{}^* = \dim \sigma - 2$, level $\sigma_1{}^* =$ level σ. Thus the level is reduced in the first case and the dimension in the second, so we proceed "double inductively": the most natural way of classifying the exceptions is by level. For each level, we first find the indecomposable exceptional classes and in each case we give a matrix characterization and also one or more technical characterizations which expedite the tedious computations needed to verify their exceptionality. Finally we see how the indecomposable exceptions are composed to give all exceptions. First we establish a useful theorem on the existence of admissible vectors outside the residual space.

1.4. *Suppose $\sigma \in \mathrm{Sp}_{2m}(V)$ has no admissible vectors in $V\backslash R$. Then one (or more) of the following hold.*

(i) $V = R$.

(ii) *σ is a hyperbolic involution.*

(iii) $F = \mathbb{F}_2$, $m = 2$, *and* $\operatorname{res} \sigma = 3$.

Proof. Suppose $V \neq R$ and σ is not a hyperbolic involution. So there exists $a \in R$ with $(\omega a, a) \neq 0$. Take $b \in V\backslash R$. Then, for $\lambda \in F$, $(\omega[b + \lambda a], b + \lambda a) = \lambda(\omega b, a) + \lambda(\omega a, b) + \lambda^2(\omega a, a)$. If $F \neq \mathbb{F}_2$, there exists $\lambda \in F$ such that the R.H.S. above is nonzero, thus $b + a$ is admissible in $V\backslash R$, contrary to hypothesis. Thus $F = \mathbb{F}_2$. According to a well-known result V can be written as an orthogonal sum of subspaces $A \perp B$ (in this case with the second component possibly vanishing) such that σ preserves A and B, $\omega_{|A}$ is nilpotent, $\omega_{|B}$ is bijective. (This is true for an isometry of an arbitrary sesquilinear space.) If $B \neq 0$ there exists $y \in B$ with $(\omega y, y) \neq 0$ since $\sigma_{|B}$ is not a hyperbolic involution. Take $x \in A\backslash R$; then $x + y$ is admissible in $V\backslash R$, not so. Thus $B = 0$ and ω is nilpotent. We claim $\operatorname{res} \sigma = 2m - 1$. If not, let $d = \sigma a$. We have $d \notin P$, so $R \not\subseteq \langle d \rangle^*$. Take T supplementary to $R \cap \langle d \rangle^*$ in $\langle d \rangle^*$. So $T \cap R = (0)$, $\dim T \geqslant 2$ since $\dim R \leqslant 2m - 2$. Hence $\dim \sigma T \geqslant 2$, so there exists a nonzero $y \in \sigma T \cap \langle a \rangle^*$, say $y = \sigma x$ with $x \in T$. Then $x + a$ is admissible in $V\backslash R$, not so. Thus $\operatorname{res} \sigma = 2m - 1$. So let $V = \{\omega^i(z)\}_{i=0}^{2m-1}$. An easy induction shows $\langle \omega^2(z), \ldots, \omega^{2m-1}(z) \rangle$ is totally degenerate, hence its dimension is $\leqslant m$, i.e., $m \leqslant 2$. But its easy to see $m \neq 1$ and so $m = 2$, $\operatorname{res} \sigma = 2m - 1 = 3$.

We also note that when $\chi(F) = 2$, if $\operatorname{res} \sigma = m + k$ with $k > m/2$, then there always exists an admissible a for σ in R, otherwise $\omega(R)$ would be

totally degenerate. (Exercise: Determine all symplectic elements σ which have no admissible vectors in R.)

We close this section with the observation that if σ of the form $\sigma_1 \perp \sigma_2$ is exceptional, then one or both components must be exceptional.

2. Levels 0 and 1

Most of the results of this section are obtained in [2, 5], so their proofs and other easy proofs are omitted.

Let $\chi(F) = 2$, m be even. H_{2m} is the matrix

$$\left[\begin{array}{c|c} & J \\ I_m & \ddots \\ & \ddots \\ & J \\ \hline & \\ & I_m \end{array}\right]$$

where I_m is the $m \times m$ identity matrix, J is the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and all other entries are zero.

2.1. *Let $\chi(F) = 2$, $\sigma \in \mathrm{Sp}_{2m}(V)$, $P \subseteq R$,* level $\sigma = 0$. *Then the following are equivalent.*

(i) *σ is exceptional.*

(ii) *σ is a hyperbolic involution.*

(iii) *σ is of class H_{2m}.*

(iv) *For all $a \in V \backslash R$, there exists a symplectic base $\mathfrak{X}$ for V with $y_1 = a$ and $m_{\mathfrak{X}}(\sigma) = H_{2m}$.*

2.2. *Let $\chi(F) = 2$. The following classes of elements of* level 0 *are not exceptional.*

(level 0, *exceptional*) $\perp$ (level 0, *not exceptional*).

Proof. Use 2.1.

2.3. *Let F be arbitrary, $\sigma \in \mathrm{Sp}_{2m}(V)$, $P \subseteq R$,* level $\sigma = 1$. *Suppose σ is indecomposable. Then one of the cases in Table I occurs.*

TABLE I

Case	dim $\omega(R) \cap P$	res σ	dim σ
1	0	2	2
2	1	3	4
3	2	4	6

Proof. (1) Let $\dim \omega(R) \cap P = 0$. Write $R = P \oplus T$ where T is a regular plane. Then it is easy to see $\omega(T)$ is a regular plane preserved by ω, hence by σ and $\operatorname{res} \sigma_{|\omega(T)} = 2$, whence the result. (2) Let $\dim \omega(R) \cap P = 1$. Let $\omega(R) \cap P = \langle x_2 \rangle$ and $\omega(x_1) = x_2$. Now $x_1 \in P + \omega(R)$ otherwise we get $x_1 \in R^* = P$. So we can take $x_1 \in \omega(R)$. Let $\omega(g_1) = x_1$ with $g_1 \in R$. Then $R = P \perp \langle x_1, g_1 \rangle$. So $(x_1, g_1) = \lambda \neq 0$. Put $y_1 = (1/\lambda) g_1$. Then $(x_1, y_1) = 1$, $(x_2, x_1) = (x_2, y_1) = 0$. Using Witt's theorem, we can extend $\{x_1, x_2, y_1\}$ to a symplectic base for V with $P = \langle x_2, x_3, \ldots, x_m \rangle$. Let $\omega y_2 = \sum_{i=1}^{m} a_i x_i + b_1 y_1$. Then $b_1 = (x_1, \sigma y_2) = -1$. Put $y_1' = y_1 - \sum_{i=3}^{m} a_i x_i$. Then $U = \langle x_1, x_2, y_1', y_2 \rangle$ is regular, preserved by σ and $\operatorname{res} \sigma_{|U} = 3$, whence the result. (3) Analogous to case 2.

From now on, the underlying field is $\mathbb{F}_2$, *except where stated.* The transvection $\tau_{x,1}$ will be denoted by τ_x.

A is the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

So $\sigma \in \mathrm{Sp}_4(V)$ is of class A iff there exists $\mathfrak{X}$ with $\sigma: x_1 \to x_1$, $x_2 \to y_2$, $y_1 \to y_1 + y_2$, $y_2 \to x_1 + x_2$.

2.4. *Let* $\sigma \in \mathrm{Sp}_4(V)$, $\operatorname{res} \sigma = 3$. (*Thus* $P \subseteq R$.) *Then the following are equivalent.*

(i) σ *is exceptional.*

(ii) $(\omega x, x) = 0$ *for all* $x \in V \backslash R$.

(iii) σ *is of class* A.

(iv) *For all admissible* a *for* σ, *there exists a symplectic base* $\mathfrak{X}$ *for* V *with* $x_2 = a$ *and* $m_{\mathfrak{X}}(\sigma) = A$.

B is the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \hline & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 1 & 0 & 1 \end{array}\right].$$

LEMMA. *Let* $\sigma \in \mathrm{Sp}_6(V)$, $\operatorname{res}\sigma = 4$, $P \subseteq R$. *Then there exists a symplectic base* $\mathfrak{X}$ *for* V *such that* $\sigma: x_1 \to x_1$, $x_2 \to x_2$, $x_3 \to x_1 + x_3$, $y_1 \to y_1 + y_3 + x_1$, $y_2 \to y_2 + x_1 + \alpha x_2 + x_3$, $y_3 \to y_3 + x_2$ *with* $\alpha \in \mathbb{F}_2$. *If* $\alpha = 0$, *there are* 16 *admissible vectors for* σ *and* σ *is of class* B. *If* $\alpha = 1$ *there are* 48 *admissible vectors for* σ *and* σ *is not exceptional.*

2.5. *Suppose* $\sigma \in \mathrm{Sp}_6(V)$, $P \subseteq R$, $\operatorname{res}\sigma = 4$. *The following are equivalent.*

(i) σ *is exceptional.*

(ii) $\omega(R) = P$ *and* σ *has* 16 *admissible vectors in* V.

(iii) σ *is of class* B.

(iv) *For all admissible* a *for* σ, *there exists a symplectic base* $\mathfrak{X}$ *for* V *with* $y_1 = a$ *and* $m_{\mathfrak{X}}(\sigma) = B$.

Proof. Use 2.3 and preceding lemma.

2.6. *The following classes of* level 1 *elements are not exceptional.*

(i) (level 1, *not exceptional*) $\perp$ (level 0).

(ii) (level 1) $\perp$ (level 0, *not exceptional*).

Proof. Use 1.4, 2.2, 2.4.

2.7. *Let* $\sigma \in \mathrm{Sp}_{2m}(V)$, $P \subseteq R$, level $\sigma = 1$. *Then* σ *is exceptional iff* σ *is of class*

(i) $A \perp H_{2m-4}$, *or*

(ii) $B \perp H_{2m-6}$,

where in each case the second component may vanish.

Proof. $\Rightarrow$: Use 2.3, 2.6, 2.4, and 2.5.

$\Leftarrow$: Use 1.3, induction on m, 2.1, 2.4, and 2.5.

3. Level 2

3.1. *Let F be arbitrary, $\sigma \in \mathrm{Sp}_{2m}(V)$, $P \subseteq R$ and* level $\sigma = 2$. *Suppose σ is indecomposable. Then one of the cases in Table II occurs. (Case 3 occurs only when $\chi(F) = 2$.)*

TABLE II

Case	dim $\omega(R) \cap P$	Properties of σ	res σ	dim σ
1	0		4	4
2	1	ω is nilpotent	5	6
3	2	$\omega^2(R) = P$, $\forall x \in R(\omega x, x) = 0$	6	8

Proof. By an analysis similar to 2.3 we show σ is of the form (level 1) $\perp$ (level 1) except in the listed cases. As a sample we will do the lengthiest case. So assume dim $\omega(R) \cap P = 2$. Let A be a plane supplementary to P in $P + \omega(R)$. (i) Suppose dim $\omega(A) \cap P = 0$. Then A is regular, otherwise there exists $r \in R\backslash(P + \omega(R))$ with $\omega(r) \in P$ and we get $P \oplus A \oplus \langle r \rangle$ totally degenerate. Now $P + \omega(R) = P + \omega(A)$ and so we get $\omega(A)$ is regular, preserved by σ and level $\sigma_{|\omega(A)} = 1$. (ii) Suppose dim $\omega(A) \cap P = 1$. Then there exists $u \in A$ and $r \in R\backslash(P + \omega(R))$ with $\omega(u)$, $\omega(r) \in P$ and so $P \oplus A \oplus \langle r \rangle$ is totally degenerate. Impossible. (iii) Suppose dim $\omega(A) \cap P = 2$. So $\omega(A) \subseteq P$, $\omega^2(R) = P$, A and thus $P + \omega(R)$ are totally degenerate. Cases:

(A) There exists $g \in R$ with $(\omega g, g) = \lambda \neq 0$. Then $g \in R\backslash(P + A)$ and $x_1 = \omega(g) \in \omega(R)\backslash P$. Since clearly $A \not\subseteq \langle g \rangle^*$, $\langle g \rangle^* \cap A$ is a line, say $\langle x_2 \rangle$. So $x_2 \notin \langle x_1 \rangle + P$ and $P + \omega(R) = P \perp \langle x_1 \rangle \perp \langle x_2 \rangle$. Put $y_1 = (1/\lambda)g$. Thus $(x_1, y_1) = 1$, $\omega(y_1) = (1/\lambda)x_1$. Let $\omega(x_1) = x_3 \in P$, $\omega(x_2) = x_4 \in P$. Extend to a symplectic base for V with $P = \langle x_3, x_4, \ldots, x_m \rangle$. Let $\omega(y_3) = b_1 y_1 + b_2 y_2 + \sum_{i=1}^{m} a_i x_i$. Then $b_1 = -1$, $b_2 = 0$. Put $y_1' = y_1 - a_1 x_2 - a_2 x_2 - \sum_{i=4}^{m} a_i x_i$, $x_1' = x_1 - \lambda a_2 x_4$. Then $U = \langle x_1', x_3, y_1', y_3 \rangle$ is regular, preserved by σ and level $\sigma_{|U} = 1$.

(B) For all $g \in R$ $(\omega g, g) = 0$. (Check this can occur only if $\chi(F) = 2$.) Write $R = P \oplus T$ with $A \subseteq T$ and T regular, four-dimensional. A is totally degenerate. So there exists a totally degenerate plane B, supplementary to A in T. Since $\omega(A) \subseteq P$, $\omega(B) \cap P = (0)$. Thus $P + \omega(R) = P + \omega(B)$. Let $B = \langle g_1, g_2 \rangle$, $\omega(g_1) = x_2$, $\omega(g_2) = x_1$. Then $R = P \oplus \langle x_1, x_2, g_1, g_2 \rangle$, $(x_1, x_2) = (x_1, g_2) = 0$. Hence $(x_1, g_1) = \lambda_1 \neq 0$. Put $y_1 = (1/\lambda_1) g_1$. Similarly $(x_2, g_2) = \lambda_2 \neq 0$ and we put $y_2 = (1/\lambda_2) g_2$. Extend to a symplectic base for V with $P = \langle x_3, \ldots, x_m \rangle$. Then we get $\omega(y_3) = -y_1 + \sum_{i=1}^{m} a_1 x_1$,

$\omega(y_4) = -y_2 + \sum_{i=1}^{m} c_i x_i$. Put $y_1' = y_1 - \sum_{i=5}^{m} a_i x_i$, $y_2' = y_2 - \sum_{i=5}^{m} c_i x_i$. Then $U = \langle x_1, x_2, x_3, x_4, y_1', y_2', y_3, y_4 \rangle$ is regular, preserved by σ and level $\sigma_{|U} = 2$. Thus $U = V$ and σ falls in case 3.

We will see there is one indecomposable exceptional class in each case (when $F = \mathbb{F}_2$). The first case is treated in [2]. Letting X be the matrix

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right],$$

the result is

3.2. *Let $\sigma \in \mathrm{Sp}_4(V)$ of residue 4. Let a be an arbitrary admissible vector for σ in V. Then the following are equivalent.*

(i) *σ is exceptional.*

(ii) *σ is of class X.*

(iii) $(\sigma^k a, a) = 0$; $k = 2, 3, 4$.

(iv) *For all admissible b, there exists $\mathfrak{X}$ with $x_1 = b$ and $m_{\mathfrak{X}}(\sigma) = X$.*

Let Y be the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right].$$

Characterizations (iii), (v) (the weak properties) and (iv), (vi) (the strong properties) of the next theorem are included to facilitate the computational aspects of the proofs for 3.4, 3.6, 4.1.

3.3. *Let $\sigma \in \mathrm{Sp}_6(V)$, $\mathrm{res}\,\sigma = 5$. Let $a \in V \backslash R$, $b \in R$ be arbitrary admissible vectors for σ. (Their existence is guaranteed by* 1.4 *and following observation.) Let $P = \langle x_1 \rangle$. Then the following are equivalent.*

(i) *σ is exceptional.*

(ii) *σ is of class Y.*

(iii) *$x_1 \in \omega(R)$, $(\sigma^k a, a) = 0$ for $k = 2, 3, 4$.*

(iv) *For all admissible* $c \in V \backslash R$, *there exists* $\mathfrak{X}$ *with* $y_1 = c$ *and* $m_{\mathfrak{X}}(\sigma) = Y$.

(v) $\omega^3 b = x_1$ *and putting* $y_3 \in \omega^{-1}(\sigma b)$, $\{x_1, b, x_1 + b + \sigma^2 b, -, \sigma b, y_3\}$ *is five-sixths of a symplectic base for* V.

(vi) *For all admissible* $d \in R$, *there exists* $\mathfrak{X}$ *with either* $x_2 = d$ *or* $x_2 = d + x_1$ *and* $m_{\mathfrak{X}}(\sigma) = Y$.

Proof. We show each assertion implies the next cyclically.

(i) $\Rightarrow$ (ii). Put $x_2 = b$, $y_2 = \sigma x_2$. So $(x_2, y_2) = 1$. Let $\sigma_1 = \tau_{\omega x_2}\sigma$, $P_1 = P_{\sigma_1} = \langle x_1, x_2 \rangle$, $R_1 = R_{\sigma_1} = P_{\sigma_1}^*$. Since $y_2 \notin P_1$, we have $R_1 \not\subseteq \langle y_2 \rangle^*$. So we can select a plane $T \subseteq R_1 \cap \langle y_2 \rangle^*$ such that $R_1 = P_1 \oplus T$. We have $P_1 \subseteq R_1$, $\operatorname{res} \sigma_1 = 4$, σ_1 exceptional. Thus, by 2.5, $\omega_1 R_1 = P_1$, where $\omega_1 = \sigma_1 - 1$. So, for all $z \in T$, $\omega_1 z \in P_1$. Thus we get $\omega(z) = a_1 x_1 + a_2 x_2 + b_2(x_2 + y_2)$ with $a_1, a_2, b_2 \in \mathbb{F}_2$. Now $0 = (x_2, z) = a_2 + b_2$. Hence $\omega(z) = a_1 x_1 + a_2 y_2$ and so $\omega(T) = \langle x_1, y_2 \rangle$. Let $x_3, y_3 \in T$ with $\omega(x_3) = x_1$, $\omega(y_3) = y_2$. T is regular, so $(x_3, y_3) = 1$ and its easy to check $x_1, x_2, x_3, -, y_2, y_3$ form part of a symplectic base for V. Extend it to $\mathfrak{X}$. Using $\sigma \in \mathrm{Sp}_6(V)$ we find $\sigma\colon y_1 \to y_1 + y_3 + c_1 x_1 + d_2 y_2$, $y_2 \to x_2 + x_3 + (1 + d_2) x_1 + f_2 y_2$ with $c_1, d_2, f_2 \in \mathbb{F}_2$. Put $y_1' = y_1 + d_2 y_3$, $x_3' = x_3 + d_2 x_1$ (still symplectic!) so σ acts thus (dropping primes): $x_1 \to x_1$, $x_2 \to y_2$, $x_3 \to x_1 + x_3$, $y_1 \to y_1 + y_3 + c_1 x_1$, $y_2 \to x_1 + x_2 + x_3 + f_2 y_2$, $y_3 \to y_2 + y_3$. Now if $f_2 = 1$, putting $c = x_2 + y_2 + y_3$ a check shows c is admissible but $\tau_{\omega c}\sigma$ is not exceptional, using 2.5. So $f_2 = 0$ and all that remains is to show if $c_1 = 0$, then σ is still of class Y. Well, if $c_1 = 0$, make the substitution $x_1' = x_1$, $x_2' = x_2 + x_1$, $x_3' = x_3$, $y_1' = y_1 + x_2 + x_3 + y_2$, $y_2' = y_2 + x_1$, $y_3' = y_3 + x_1 + x_3$. Then a check shows $\mathfrak{X}' = \{x_i', y_i'\}_{i=1}^3$ is a symplectic base for V and $m_{\mathfrak{X}'}(\sigma) = Y$. (ii) $\Rightarrow$ (iii). Routine, after computing admissible a's. (iii) $\Rightarrow$ (iv). Put $y_1 = a$, $x_3 \in \omega^{-1}(x_1)$. So $x_3 \in R$ and $(\sigma^k y_1, y_1) = 0$ for $k = 2, 3, 4$ by hypothesis. Put $y_2 = \omega^2 y_1$, $y_3 = x_1 + \omega y_1$, $x_2 = \sigma y_2 + x_1 + x_3$. Then $\mathfrak{X} = \{x_i, y_i\}_{i=1}^3$ is a symplectic base for V. Using $\sigma \in \mathrm{Sp}_6(V)$ and $(\sigma^4 y_1, y_1) = 0$ we get $\sigma\colon x_2 \to y_2$ and $m_{\mathfrak{X}}(\sigma) = Y$. Now the (ii) $\Rightarrow$ (iii) proof shows a can be replaced by *any* admissible $c \in V \backslash R$ and the result follows. (iv) $\Rightarrow$ (v). Routine, after computing admissible b's. (v) $\Rightarrow$ (vi). According to hypothesis, form a symplectic base $\mathfrak{X}$ with $x_2 = b$, $x_3 = x_1 + b + \sigma^2 b$, $y_2 = \sigma b$, $y_3 \in \omega^{-1}(\sigma b)$. Using $\sigma \in \mathrm{Sp}_6(V)$ and $\omega^3 b = x_1$ we get $\sigma\colon y_1 \to y_1 + y_3 + a_1 x_1$. Now, if $a_1 = 1$ then $m_{\mathfrak{X}}(\sigma) = Y$ and $x_2 = b$, and if $a_1 = 0$ making the substitution as in the (i) $\Rightarrow$ (ii) proof (which includes $x_2' = x_2 + x_1$) we get a symplectic base $\mathfrak{X}'$ such that $m_{\mathfrak{X}'}(\sigma) = Y$, $x_2' = b + x_1$. As before, an arbitrary admissible $d \in R$ can play the rôle of b and we are done. (vi) $\Rightarrow$ (i). First note (vi) implies (ii) so (iv) holds, and thus it is easy to verify using 2.5 and 3.2 that $\tau_{\omega c}\sigma$ is exceptional for all admissible c for σ in V.

Z is the matrix

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 1 & 0 & 1 & 0 \\ & & & & 0 & 1 & 0 & 1 \end{array}\right].$$

3.4. *Let $\sigma \in \mathrm{Sp}_8(V)$, $P \subseteq R$, $\operatorname{res} \sigma = 6$. The following are equivalent.*

(i) *σ is exceptional and indecomposable.*

(ii) *$(\omega r, r) = 0$ for all $r \in R$.*

(iii) *σ is of class Z.*

(iv) *For all admissible a for σ, there exists $\mathfrak{X}$ with $a = y_1$ or $a = y_1 + x_2$ and $m_{\mathfrak{X}}(\sigma) = Z$.*

Proof. (i) $\Rightarrow$ (ii). This follows from 3.1.

(ii) $\Rightarrow$ (iii). Let $R = P \oplus T$ with T four-dimensional, regular and let $\omega(R) = P \oplus A$ with $A \subseteq T$. A check shows σ must be indecomposable and fall in case 3 of Table 2. Hence $\omega(R)$ is totally degenerate and we can write $T = A \oplus B$ with $B = \langle y_3, y_4'' \rangle$ and $(y_3, y_4'') = 0$. Let $P = \langle x_1, x_2 \rangle$ and put $x_3 = \omega(y_4'') + x_1$, $x_4 = \omega(y_3) + x_2$. This defines part of a symplectic base; extend to $\{x_1, x_2, x_3, x_4, y_1'', y_2, y_3, y_4''\}$. Say $\sigma: y_1'' \to y_1'' + \sum a_1 x_1 + b_3 y_3 + b_4 y_4''$, $y_2 \to y_2 + \sum c_1 x_1 + d_3 y_3 + d_4 y_4''$. Putting $y_1 = y_1'' + a_2 x_4$, $y_4 = y_4'' + a_2 x_1$ we get a symplectic base $\mathfrak{X}$ and since $\sigma \in \mathrm{Sp}_8(V)$ we have $\sigma: y_1 \to y_1 + y_3 + a_1 x_1$, $y_2 \to y_2 + y_4 + c_2 x_2$. Now consider the substitution $x_3' = x_3 + x_2$, $y_1' = y_1 + x_2$, $y_2' = y_2 + x_1 + y_3$, $y_4' = y_4 + x_4$. Then $\mathfrak{X}' = \{x_1, x_2, x_3', x_4, y_1', y_2', y_3, y_4'\}$ is a symplectic base and $m_{\mathfrak{X}'}(\sigma)$ is the same as $m_{\mathfrak{X}}(\sigma)$ but with c_2 replaced by $c_2 + 1$. Thus $c_2 = 0$ and $c_2 = 1$ give the same class. By symmetry between even and odd subscripts $a_2 = 0$ and $a_2 = 1$ give the same class and so σ is of class Z.

(iii) $\Rightarrow$ (iv). First check $(\omega^2 a, a) = (\omega^3 a, a) = 0$, $\omega^3 a \in P$ for all admissible a and $(\omega r, r) = 0$ for all $r \in R$. Take $x_1 \in P$ with $(x_1, a) = 1$ (taking fixed a). Put $y_1 = a$, $y_3 = \omega(a) + x_1 \in R$, let $P \cap \langle y_1 \rangle^* = \langle x_2 \rangle$, put $x_4 = \omega(y_3) + x_2$. Take x_3 so that $\omega(x_3) = x_1$ and $(x_3, y_1) = 0$ (possible since $(x_1, y_1) = 1$ and $\omega(x_1) = 0$). Then $\omega(x_4) = \omega^3 a \in P \cap \langle a \rangle^*$ and so $\omega(x_4) = x_2$. Extend to symplectic base $\mathfrak{X}$. Then a computation using the substitution in (ii) $\Rightarrow$ (iii) proof, if necessary, yields the stated result.

(iv) $\Rightarrow$ (i). σ is of class Z and so indecomposable (check). And it is easy by 3.3 to show σ is exceptional.

3.5. *The following classes of* level 2 *elements are not exceptional.*

(i) (level 2, *not exceptional*) $\perp$ (level 0, *exceptional*).

(ii) (level 2, *exceptional*) $\perp$ (level 0, *not exceptional*).

(iii) (level 1) $\perp$ (level 1).

3.6. *Let* $\sigma \in \mathrm{Sp}_{2m}(V)$, $P \subseteq R$, level $\sigma = 2$. *Then* σ *is exceptional iff* σ *is of class*

(i) $X \perp H_{2m-4}$, *or*

(ii) $Y \perp H_{2m-6}$, *or*

(iii) $Z \perp H_{2m-8}$,

where, in each case the second component may vanish.

Proof. $\Rightarrow$: Use 3.1, 2.6, 3.5 and then 3.2, 3.3, 3.4.

$\Leftarrow$: Induction on m, using 1.3, 2.1 and 3.2, 3.3, 3.4.

4. Level 3

Here we show there are no further exceptions. First we consider the smallest dimension for level 3, viz. residue 6 in dimension 6 and prove

4.1. *There are no exceptions of residue* 6 *in* $\mathrm{Sp}_6(V)$.

Proof. Suppose for a contradiction $\sigma \in \mathrm{Sp}_6(V)$, res $\sigma = 6$ and σ is exceptional. Then there exists an admissible vector x_1, say, for σ in V and so $\sigma_1 = \tau_{\omega x_1}\sigma$ is exceptional. Hence, by 3.3, σ_1 is of class Y and there exists $\mathfrak{X} = (x_i, y_i)_{i=1}^3$ with $m_{\mathfrak{X}}(\sigma_1) = Y$. Let $y_1' = \sigma x_1 = y_1 + a_1x_1 + a_2x_2 + a_3x_3 + b_2y_2 + b_3y_3$ and put $x_2' = x_2 + b_2x_1$, $x_3' = x_3 + b_3x_1$, $y_2' = y_2 + a_2x_1$, $y_3' = y_3 + a_3x_1 + a_2x_3'$. Thus $\mathfrak{X}' = \{x_1, x_2', x_3', y_1', y_2', y_3'\}$ is a symplectic base for V. Computing the action of σ on this base, using $\sigma z = \sigma_1 z + (\omega z, x_1)[y_1' + x_1]$, we find σ acts as follows:

$$
\begin{aligned}
&x_1 \to y_1', \qquad x_2' \to \alpha y_1' + y_2', \qquad x_3' \to x_3' + y_1',\\
&y_1' \to x_1 + \alpha x_2' + \alpha x_3' + t y_1' + (1+\beta) y_2' + y_3',\\
&y_2' \to x_2' + x_3' + \beta y_1', \qquad y_3' \to y_2' + y_3',
\end{aligned}
$$

where $\alpha = a_2 + b_2$, $\beta = a_2 + d_2$, $t \in \mathbb{F}_2$. Now we have x_2', y_2' are each admissible for σ, hence $\tau_{\omega x_2'}\sigma$, $\tau_{\omega y_2'}\sigma$ are exceptional which yields by 3.3

$t_1 + \alpha\beta = 0$ and $\alpha = 0$, thus $t_1 = 0$. But $a = x_1 + x_2 + x_3$ is admissible for σ and 3.3 now shows $\tau_{\omega a}\sigma$ is not exceptional [take cases $\beta = 0$, 1, separately]. Thus the original σ is not exceptional and we get the desired conclusion.

An easy induction, principally using 1.4, now shows there are no exceptions of level 3 or higher.

5. Summary

In 2.1, 2.7, and 3.6 we have determined all exceptions with $P \subseteq R$. Since the hyperbolic involutions consist of elements of the form $1 \perp H_{2k}$ (where either component may vanish) we obtain in view of 1.2 and 1.1 the main result. Recall $F = \mathbb{F}_2$.

Theorem 5.1. *The set of exceptional symplectic elements over $\mathbb{F}_2$ which cannot be expressed as a product of* res σ *symplectic transvections is the following. σ of the form $\psi \perp \phi$ where ψ is of class H_4, A, B, X, Y, or Z and ϕ is a hyperbolic involution, possibly vanishing. Any such exception σ can be expressed as the product of* res $\sigma + 1$ *symplectic transvections.*

Remarks. Two exceptions σ_1, σ_2 in the same symplectic group are conjugate iff ψ_1, ψ_2 are of the same class and ϕ_1, ϕ_2 have the same residue and dimension. The distinct conjugacy classes of indecomposable exceptional symplectic elements are precisely the classes H_4, A, B, X, Y, and Z. Any exceptional element is the orthogonal sum of an indecomposable exception and a (possibly vanishing) hyperbolic involution, and the orthogonal sum of two exceptions is not exceptional unless at least one of them is a hyperbolic involution.

6. The Orthogonal Case

We follow the notation in [2]. So Q is a nondegenerate, nondefective quadratic form on the n-dimensional vector space V over a field F of characteristic 2 with associated bilinear form $(\ ,\ )$ which is necessarily alternating. Thus n is even, $n = 2m$ say, and $O_n(V) \subseteq \mathrm{Sp}_n(V)$. The orthogonal transvections, i.e., the transvections in $O_n(V)$ are all those of the form $\tau_{a,[Q(a)]^{-1}}$ with $a \in V$, $Q(a) \neq 0$. The orthogonal transvections generate $O_n(V)$ except when $n = 4$, $F = \mathbb{F}_2$ and the index of Q is 2 [3, p. 42]. We will call an element $\sigma \in O_n(V)$ an *orthogonal exception* iff it cannot be expressed as a product of res σ orthogonal transvections. The following two facts are easily proved.

(i) Let τ be a transvection in $O_n(V)$, $\sigma \in O_n(V)$. Then $\operatorname{res} \tau\sigma = \operatorname{res} \sigma - 1$ if $R_\tau \subseteq R_\sigma$ and $\operatorname{res} \tau\sigma = \operatorname{res} \sigma + 1$ if $R_\tau \not\subseteq R_\sigma$.

(ii) Let τ be a transvection in $\mathrm{Sp}_n(V)$, $\sigma \in O_n(V)$. If $\operatorname{res} \tau\sigma < \operatorname{res} \sigma$, then $\tau \in O_n(V)$.

It follows easily from (i) that the number of factors in any expression of $\sigma \in O_n(V)$ as a product of orthogonal transvections is congruent to $\operatorname{res} \sigma \pmod 2$ and from (ii) that $\sigma \in O_n(V)$ is an orthogonal exception iff it is a (symplectic) exception. We note without proof that each class of symplectic exception can occur in an appropriate orthogonal group so the question remaining is: What is the minimal number of factors for orthogonal exceptions? The answer is, excluding the bad situation $n = 4$, $F = \mathbb{F}_2$, index $Q = 2$ referred to above, $\operatorname{res} \sigma + 2$ if $F \neq \mathbb{F}_2$ or if $F = \mathbb{F}_2$ but σ not of class H_{2m}, $\operatorname{res} \sigma \mid 4$ if $F = \mathbb{F}_2$ and σ is of class H_{2m}. The proofs are easy; e.g., for σ of class B and $F = \mathbb{F}_2$ the only danger would be that $\tau_{a,[Q(a)]^{-1}}$ might be of class Y for all $a \in V \backslash R$ with $Q(a) \neq 0$, in which case $\operatorname{res} \sigma + 2$ would be insufficient but this can be checked not so.

The possibility that Dieudonné's version [2, p. 164] of the main result, Theorem 5.1, of this paper is incomplete was first noted by Professor O. T. O'Meara during lectures at the University of Notre Dame.

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