

WILF CLASSES FOR WEAK ASCENT SEQUENCES AVOIDING A PAIR OR TRIPLE OF LENGTH-3 PATTERNS

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ABSTRACT. A *weak ascent sequence* is a word $\pi = \pi_1\pi_2\cdots\pi_n$ over the set of nonnegative integers such that $\pi_1 = 0$ and $\pi_i \leq 1 + \text{weak_asc}(\pi_1\pi_2\cdots\pi_{i-1})$ for $i = 2, \dots, n$, where $\text{weak_asc}(\pi_1\pi_2\cdots\pi_m)$ is the number of *weak ascents* in the word $\pi_1\pi_2\cdots\pi_m$, that is, the number of positions j such that $\pi_j \leq \pi_{j+1}$. Here we obtain some enumerative results for weak ascent sequences avoiding a set of two or three 3-letter patterns, leading to a conjecture for the number of Wilf equivalence classes for weak ascent sequences, called *A-Wilf equivalence classes*, avoiding a pair (respectively, triple) of 3-letter patterns. The main tool is the use of generating trees. Some cases are treated using bijective methods.

1. INTRODUCTION

An *ascent* in a sequence $s_1s_2\cdots s_m$ is an index $1 \leq j \leq m-1$ such that $s_j < s_{j+1}$. An *ascent sequence* $a_1a_2\cdots a_n$ is one consisting of non-negative integers satisfying $a_1 = 0$ and for all i with $1 < i \leq n$, $a_i \leq \text{asc}(a_1a_2\cdots a_{i-1}) + 1$, where $\text{asc}(a_1a_2\cdots a_k)$ is the number of ascents in the sequence $a_1a_2\cdots a_k$. For example, the sequence 01013224 is an ascent sequence, whereas 01003 is not. Bousquet-Mélou, Claesson, Dukes, and Kitaev [3] connected ascent sequences to $(2+2)$ -free posets. Since then, ascent sequences have been considered in a series of papers where connections to many other combinatorial structures have been found.

The concept of ascent sequence has been extended to weak ascent sequences, where Bényi, Claesson, and Dukes [1] found a relation between weak ascent sequences and permutations avoiding a length-4 bivincular pattern, restricted upper-triangular binary matrices, and factorial posets that are weakly $(3+1)$ -free. A *weak ascent sequence* is a word $\pi = \pi_1\pi_2\cdots\pi_n$ over the set of non negative integers such that $\pi_1 = 0$ and $\pi_i \leq 1 + \text{weak_asc}(\pi_1\pi_2\cdots\pi_{i-1})$ for $i = 2, \dots, n$, where $\text{weak_asc}(\pi_1\pi_2\cdots\pi_m)$ is the number of *weak ascents* in the word $\pi_1\pi_2\cdots\pi_m$, that is, the number of positions j such that $\pi_j \leq \pi_{j+1}$. We denote the set of all weak ascent sequences of length n by $\mathcal{A}_n$. For example, $\mathcal{A}_3 = \{000, 001, 002, 010, 011, 012\}$ while $\{0010, 0011, 0012, 0013\} \subset \mathcal{A}_4$, $0014 \notin \mathcal{A}_4$. A *pattern* π is a word containing each of the letters in $\{0, 1, \dots, \ell-1\}$ at least once for some integer $\ell \geq 1$. A weak ascent sequence $w = w_1w_2\cdots w_n \in \mathcal{A}_n$ *contains* a pattern $\pi = \pi_1\cdots\pi_k$ if there is a subsequence $w_{i_1}w_{i_2}\cdots w_{i_k}$ of w , $i_1 < i_2 < \cdots < i_k$, which is order-isomorphic to π . We say that w *avoids* the pattern π if w does not contain any occurrence of π . For example, the weak ascent sequence $00203413 \in \mathcal{A}_8$ contains the pattern 011 and avoids the pattern 321. We denote the set of weak ascent sequences of length n that avoid π by $\mathcal{A}_n(\pi)$. Further, for a set of patterns B , we set $\mathcal{A}_n(B) = \bigcap_{\pi \in B} \mathcal{A}_n(\pi)$.

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The goal of this work is to study weak ascent sequences avoiding a set of two or three length-3 patterns. This study is analogous to the one conducted by Duncan and Steingrímsson on ascent sequences [4, 5]. More precisely, we say that two sets of patterns P and Q are A -Wilf equivalent, denoted $P \stackrel{a}{\sim} Q$, if $|\mathcal{A}_n(P)| = |\mathcal{A}_n(Q)|$ for every $n \geq 0$. Let WAS_k denote the number of A -Wilf equivalence classes for weak ascent sequences avoiding k distinct patterns of length three. The main result of this paper can be formulated briefly as follows.

Theorem 1.1. *We have $59 \leq WAS_2 \leq 61$ and $150 \leq WAS_3 \leq 155$.*

Conjecture 1.2. *We have $WAS_2 = 59$ and $WAS_3 = 150$.*

2. GENERATING TREES

We apply here the generating tree method introduced by West [8]. Given a fixed set B of patterns of length at least 2, let $\mathcal{A}(B)$ denote the set of all non-empty weak ascent sequences that avoid B . We define the generating tree $\mathcal{T}(B)$ to be the plain tree with root 0 as follows, see [2]. Starting with this root which stays at level 1, we construct the remainder of the nodes of the tree $\mathcal{T}(B)$ recursively as follows. The children of $x_1 \cdots x_n \in \mathcal{A}_n(B)$ are those sequences from the set

$$\{x_1 \cdots x_{n+1} \mid x_{n+1} = 0, 1, \dots, \#\text{weak_asc}(x_1 \cdots x_n) + 1\}$$

that avoid the patterns in B .

We define an equivalence relation on the nodes of $\mathcal{T}(B)$. Let $\mathcal{T}(B; w)$ be the subtree consisting of the weak ascent sequence w as the root and its descendants in $\mathcal{T}(B)$. We say that w is *equivalent* to w' if and only if $\mathcal{T}(B; w) \cong \mathcal{T}(B; w')$ (in the sense of plain trees). Let $\mathcal{T}'(B)$ be the same tree as $\mathcal{T}(B)$, where we replace each node w by the first node $w' \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; w) \cong \mathcal{T}(B; w')$. From now, we consider $\mathcal{T}'(B)$ and $\mathcal{T}(B)$ identical.

For each node w in $\mathcal{T}(B)$, we define $W_w(x)$ be the generating function for the number nodes at level n in the subtree $\mathcal{T}(B; w)$ of $\mathcal{T}(B)$, where the root stay at level 1, namely,

$$W_w(x) = \sum_{n \geq 1} |\{\text{nodes at level } n \text{ in the subtree } \mathcal{T}(B; w) \text{ of } \mathcal{T}(B)\}| \cdot x^n.$$

The basic outline of the generating tree method is the following.

- (1) Let B be any set of patterns and let D be any positive number (here we use $D = 5$).
- (2) We find the first D levels of the generating tree $\mathcal{T}(B)$.
- (3) By (2), we guess all the succession rules of $\mathcal{T}(B)$.
- (4) Based on (3), we try to prove these succession rules (for instance, see Example 2.1). If we fail, then we increase D by 1 and go back to Step (2). Otherwise, the succession rules of the generating tree $\mathcal{T}(B)$ are found.
- (5) Each rule $v \rightsquigarrow w^{(1)}, \dots, w^{(s)}$ of $\mathcal{T}(B)$, we translate to an equation as $W_v(x) = x + x \sum_{j=1}^s W_{w^{(j)}}(x)$. Then, we try to solve the obtained system of the equations for $W_0(x) = \sum_{n \geq 1} |\mathcal{A}_n(B)| x^n$.

Example 2.1. Let $B = \{001, 101\}$. Based on the first 5 levels of the generating tree $\mathcal{T}(B)$, we obtain the following rules

$$\begin{array}{ll}
0 \rightsquigarrow 00, 01, & 00 \rightsquigarrow 000, \\
01 \rightsquigarrow 00, 011, 012, & 000 \rightsquigarrow 0000, \\
011 \rightsquigarrow 000, 0111, & 012 \rightsquigarrow 000, 011, 0122, 0123, \\
0000 \rightsquigarrow 00000, & 0111 \rightsquigarrow 0000, 01111, \\
0122 \rightsquigarrow 0000, 0111, 01222, & 0123 \rightsquigarrow 0000, 0111, 0122, 01233, 01234, \\
00000 \rightsquigarrow 000000, & 01111 \rightsquigarrow 00000, 011111, \\
01222 \rightsquigarrow 00000, 01111, 012222, & 01233 \rightsquigarrow 00000, 01111, 01222, 012333, \\
01234 \rightsquigarrow 00000, 01111, 01222, 01233, 012344, 012345.
\end{array}$$

Thus, we guess that the generating tree $\mathcal{T}(B)$ has the following rules

$$\begin{array}{ll}
a_1 \rightsquigarrow a_2, b_1, & a_m \rightsquigarrow a_{m+1}, \\
b_m \rightsquigarrow a_{m+1}, c_{1,m}, \dots, c_{m-1,2}, c_{m,2}, b_{m+1}, & c_{m,j} \rightsquigarrow a_{m+j}, c_{1,m+j-1}, \dots, c_{m-1,j+1}, c_{m,j+1},
\end{array}$$

where $a_m = 0^m$, $b_m = 01 \dots m$, $c_{m,j} = 01 \dots (m-1)m^j$. This can be explained as follows:

- The only children of a_1 are a_2 and b_1 , so $a_1 \rightsquigarrow a_2, b_1$ holds.
- The only child of a_m with $m \geq 2$ is a_{m+1} (remember, we avoid B), so $a_m \rightsquigarrow a_{m+1}$ holds.
- The children of b_m are $b_m j$ with $j = 0, \dots, m+1$. Note that $\mathcal{T}(B; b_m 0) \cong \mathcal{T}(B; a_{m+1})$, $b_m(m+1) = b_{m+1}$, $b_m m = c_{m,2}$, $\mathcal{T}(B, b_m(m-1)) \cong \mathcal{T}(B; c_{m-1,2})$, and $\mathcal{T}(B, b_m j) \cong \mathcal{T}(B; 01 \dots (j-1)j^{m+1-j}) = \mathcal{T}(B; c_{j,m+1-j})$ with $j = 1, 2, \dots, m-2$. So, $b_m \rightsquigarrow a_{m+1}, c_{1,m}, \dots, c_{m-1,2}, c_{m,2}, b_{m+1}$ holds.
- Similarly, the last two rules are holds.

Let $A_m(x) = W_{a_m}(x)$, $B_m(x) = W_{b_m}(x)$, and $C_{m,j}(x) = W_{c_{m,j}}(x)$. Thus, the rules can be translated to

$$\begin{aligned}
A_1(x) &= x + xA_2(x) + xB_1(x), \\
A_m(x) &= x + xA_{m+1}(x), \\
B_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^{m-1} C_{i,m+1-i}(x) + xC_{m,2}(x) + xB_{m+1}, \\
C_{m,j}(x) &= x + xA_{m+j}(x) + x \sum_{i=1}^{m-1} C_{i,m+j-i}(x) + xC_{m,j+1}(x).
\end{aligned}$$

By induction on m , we have $A_m(x) = \frac{x}{1-x}$ and $C_{m,j}(x) = \frac{x}{(1-x)^{m+1}}$. Thus,

$$B_m(x) = \frac{x}{1-x} + x \sum_{i=1}^{m-1} \frac{x}{(1-x)^{i+1}} + \frac{x^2}{(1-x)^{m+1}} + xB_{m+1}(x).$$

Hence, $B_1(x) = \frac{x}{(1-x)(1-2x)}$, which implies that

$$W_0(x) = A_1(x) = x + \frac{x^2}{1-x} + \frac{x^2}{(1-x)(1-2x)} = \frac{x}{1-2x}.$$

3. AVOIDING A PAIR OF PATTERNS OF LENGTH 3

A *candidate class* is a maximal subset C of the set of all pairs (in next section, triples) of patterns of length 3 such that for any $P, P' \in C$, $|\mathcal{A}_n(P)| = |\mathcal{A}_n(P')|$, for all $n = 1, 2, \dots, 11$. A candidate class is called *trivial* if it contains exactly one pair of patterns, otherwise, it is called *nontrivial*. Thus a trivial class certainly constitutes a (singleton) A -Wilf equivalence class. By Table 3 there are exactly 48 trivial and 11 nontrivial candidate classes. The aim of this section is to show that the 11 nontrivial candidate classes of pairs of length 3 patterns are indeed 11 A -Wilf equivalences. Table 1 describes all these 11 nontrivial candidate classes including the corresponding succession rules of the generating trees and explicit formulas (if they exist) for the generating function. In classes 15 and 21 bijective methods are used. The two open cases, classes 41 and 43, are in blue and indicated by the double question marks.

Table 1: Rules of generating trees for weak ascent sequences avoiding a pair of length-3 patterns.

Beginning of Table 1			
Class	B pair	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
1	$\{001, 010\}$ $\{001, 011\}$ $\{001, 012\}$	$01 \dots m \rightsquigarrow 0^{m+2}, 01 \dots (m+1); 0^m \rightsquigarrow 0^{m+1}$ $01^m \rightsquigarrow 0^{m+2}, 01^{m+1}; 0^m \rightsquigarrow 0^{m+1}$	$\frac{x}{(1-x)^2}$
5	$\{000, 011\}$ $\{001, 021\}$ $\{001, 120\}$ $\{001, 110\}$	$0 \rightsquigarrow a_0, 01; a_m \rightsquigarrow a_{m+1}, b_{m+2}; 01 \rightsquigarrow 010, b_1;$ $b_m \rightsquigarrow c_m, b_{m+1}; c_m \rightsquigarrow c_{m+1}; 010 \rightsquigarrow c_2$, where $a_m = 0012 \dots m, b_m = 0012 \dots (m-2)m,$ $c_m = 0012 \dots (m-2)m(m-1)$ $a_1 \rightsquigarrow a_2, c_1; a_m \rightsquigarrow a_{m+1}; c_m \rightsquigarrow a_{m+1}, b_{m+1}, c_{m+1};$ $b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m, b_m = 01^m,$ $c_m = 01 \dots m$ $a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow (a_m)^m, a_{m+1}, b_{m+1},$ where $a_m = 0^m, b_m = 01 \dots m$	$\frac{x}{1-x} + \frac{x^2}{(1-x)^3}$
8	$\{001, 101\}$ $\{001, 102\}$ $\{001, 201\}$ $\{010, 012\}$ $\{011, 012\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1};$ $b_m \rightsquigarrow a_{m+1}, c_{1,m}, \dots, c_{m-1,2}, c_{m,2}, b_{m+1};$ $c_{m,j} \rightsquigarrow a_{m+j}, c_{1,m+j-1}, \dots, c_{m-1,j+1}, c_{m,j+1}$, where $a_m = 0^m, b_m = 01 \dots m, c_{m,j} = 01 \dots (m-1)m^j$ $a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j-1}, b_{m+1,j}$, where $a_m = 0^m,$ $b_{m,j} = a_m j$	Example 2.1, Section 3.1; $\frac{x}{1-2x}$
11	$\{012, 021\}$ $\{012, 110\}$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow b_m, b_{m+1}$, where $a_m = 0^m,$ $b_m = a_m 1$ $a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j}, c_m; c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m, b_{m,j} = a_m j, c_m = a_m 11$	Section 3.2; $\frac{x(2x^2-2x+1)}{(1-x)^2(1-2x)}$
13	$\{012, 201\}$ $\{012, 210\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, c_{m,2}, \dots, c_{m,j}, b_{m+1,j};$ $c_{m,j} \rightsquigarrow d_m, c_{m,2}, \dots, c_{m,j-1}, c_{m+1,j}; d_m \rightsquigarrow d_{m+1},$ where $a_m = 0^m, b_{m,j} = a_m j, c_{m,j} = a_m m(m-1),$ $d_m = a_m 210$ $a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,j}, (d_m)^{j-1}, b_{m+1,j}; d_m \rightsquigarrow d_{m+1}$, where $a_m = 0^m, b_{m,j} = a_m j, d_m = a_m 21$	Section 3.3; $\frac{x(2x^4-5x^3+7x^2-4x+1)}{(1-x)^4(1-2x)}$
15	$\{011, 102\}$ $\{012, 102\}$ $\{012, 120\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j}, b_{m+1,j}$, where $a_m = 0^m,$ $b_{m,j} = a_m j;$	See Section 3.4 See [2], $\frac{x(1-x)}{1-3x+x^2}$
21	$\{010, 021\}$ $\{011, 201\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m+1,j}, \dots, b_{m+1,m+1}$, where $a_m = 0^m,$ $b_{m,j} = a_m j$	Section 3.5, $\frac{1-\sqrt{1-4x}}{2x} - 1$
24	$\{011, 101\}$ $\{011, 110\}$	$\sim \{011\}$	See [2]

Continuation of Table 1			
Class	B pair	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
41	$\{102,110\}$ $\{102,120\}$	??	See A279561
43	$\{021,201\}$ $\{021,210\}$ $\{101,102\}$	$\sim \{021\}$??	See [2]
57	$\{100,201\}$ $\{101,201\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow c_{m,j,0}, \dots, c_{m,j,j-1}, b_{m+1,j}, \dots, b_{m+1,m+1};$ $c_{m,j,i} \rightsquigarrow c_{m,j,0}, \dots, c_{m,j,i-1},$ $c_{m+1,j,i}, c_{m+1,j+1,i+1}, \dots, c_{m+1,m+1,m+1+i-j},$ where $a_m = 0^m, b_{m,j} = a_m j, c_{m,j,i} = a_m j i$	
End of Table 1			

3.1. **Class 8.** In Example 2.1 we showed that the generating functions for the number weak ascent sequences of length n that avoid $\{001, 101\}$ is given by $\frac{x}{1-2x}$. Thus, it remains to study the case $B = \{010, 012\}$. Let $A_m(x)$ (respectively, $B_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(B)$ with a root a_m (respectively, b_m). Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x),$$

$$B_{m,j}(x) = x + x \sum_{i=1}^{j-1} B_{m,i}(x) + xB_{m+1,j}.$$

By induction on j , we have $B_{m,j}(x) = \frac{x}{(1-x)^j}$. Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m \frac{x}{(1-x)^i},$$

Hence, $A_1(x) = \frac{x}{1-2x}$.

3.2. **Class 11. Case $\{012,021\}$** Let $A_m(x)$ (respectively, $B_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 021)$ with a root a_m (respectively, b_m). Thus,

$$A_m(x) = x + xA_{m+1}(x) + mxB_m(x),$$

$$B_m(x) = x + xB_m(x) + xB_{m+1}(x).$$

By induction on m , we have $B_m(x) = \frac{x^2}{1-2x}$. Thus,

$$A_m(x) = x + xA_{m+1}(x) + \frac{mx^2}{1-2x}.$$

Hence, $A_1(x) = \frac{x(2x^2-2x+1)}{(1-x)^2(1-2x)}$.

Case $\{012,110\}$ Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 110)$ with a root a_m (respectively, $b_{m,j}$, c_m). Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x),$$

$$B_{m,j}(x) = x + x \sum_{i=1}^j B_{m,i}(x) + xC_m(x),$$

$$C_m(x) = x + xC_{m+1}(x).$$

By induction on m , $C_m(x) = \frac{x}{1-x}$. So, by induction on j , we have that $B_{m,j}(x) = \frac{x}{(1-x)^{j+1}}$. Hence,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m \frac{x}{(1-x)^{i+1}},$$

which implies that $A_1(x) = \frac{x(2x^2-2x+1)}{(1-x)^2(1-2x)}$.

3.3. Class 13. Case {012,201} Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 201)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, d_m). Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x),$$

$$B_{m,j}(x) = x + xB_{m,1}(x) + x \sum_{i=2}^j C_{m,i}(x) + xB_{m+1,j}(x),$$

$$C_{m,j}(x) = x + xD_m(x) + x \sum_{i=2}^{j-1} C_{m,i}(x) + xC_{m+1,j}(x),$$

$$D_m(x) = x + xD_{m+1}(x).$$

By induction on m, j , we have that $D_m(x) = \frac{x}{1-x}$, $C_{m,j}(x) = \frac{x}{(1-x)^j}$, and $B_{m,j}(x) = \frac{x}{(1-x)^{j+1}} + \frac{x^3}{(1-x)^2(1-2x)}$. Hence,

$$A_m(x) = x + xA_{m+1}(x) + \sum_{j=1}^m \frac{x^2}{(1-x)^{j+1}} + \frac{mx^4}{(1-x)^2(1-2x)},$$

which implies that

$$A_1(x) = \frac{x(2x^4 - 5x^3 + 7x^2 - 4x + 1)}{(1-x)^4(1-2x)}.$$

Case {012,201} Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $D_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 210)$ with a root a_m (respectively, $b_{m,j}$, d_m). Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x),$$

$$B_{m,j}(x) = x + xB_{m,j}(x) + (j-1)xD_m(x) + xB_{m+1,j}(x),$$

$$D_m(x) = x + xD_{m+1}(x).$$

By induction on m, j , we have that $D_m(x) = \frac{x}{1-x}$ and $B_{m,j}(x) = \frac{x(jx-2x+1)}{(1-x)(1-2x)}$. Hence,

$$A_m(x) = x + xA_{m+1} + \sum_{j=1}^m \frac{x^2(jx-2x+1)}{(1-x)(1-2x)},$$

which implies that

$$A_1(x) = \frac{x(2x^4 - 5x^3 + 7x^2 - 4x + 1)}{(1-x)^4(1-2x)}.$$

3.4. Class 15. Note that $\{012, 102\} \stackrel{a}{\sim} \{012\} \stackrel{a}{\sim} \{012, 120\}$. Thus, by [2], we have that the generating function for the number of weak ascent sequences of length n that avoid $\{012, \tau\}$ is given by $\frac{x(1-x)}{1-3x+x^2}$, that is, the number of weak ascent sequences of length n that avoid $\{012, \tau\}$ is given by F_{2n-1} , where F_n is the n th Fibonacci number (defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$). So, it remains to show that $|\mathcal{A}_n(\{011, 102\})| = F_{2n-1}$, for all $n \geq 1$.

The following lemma is immediate from the definition of a weak ascent sequence.

Lemma 3.1. *The maximum possible entry in position j in a weak ascent sequence is $j - 1$ for $j \geq 1$. $\square$*

For a weak ascent sequence, let m denote the number of 0s it contains and let r denote the number of its descents. Suppose $w \in \mathcal{A}_n(011, 102)$. The nonzero entries of w are all distinct to avoid 011; say they form the *positive-support* set of w , denoted $S = S(w)$. Then $1 \leq m \leq n$ and $|S| = n - m$. Say $A = \max(S)$ (when $S \neq \emptyset$). The first descent top (if there is one) must be A to avoid a 102 pattern. Hence, $r \leq m - 1$ for suppose w begins with $p \geq 1$ 0s followed by $q \geq 0$ increasing positive entries followed by A , the first descent top. Then $A \leq p + q$ by the weak ascent condition. This implies there are at most p descent tops and if there are p descent tops, then another 0 is a descent bottom. In any case, $(\# \text{ descents}) \leq (\# \text{ 0s}) - 1$ and $r \leq m - 1$. Also, there cannot be more descents than elements of S , so $r \leq n - m$.

Now, A occurs at or before position $n - r$ to allow space for r descents following A . So $A \leq n - r - 1$ by Lemma 3.1, and $A \geq n - m$ since the elements of S are distinct. Hence, $A = n - m + j$ for some $j \in \{0, 1, \dots, m - 1 - r\}$ and A occurs at or after position $n - m + j + 1$ by Lemma 3.1.

Say $d_1 < d_2 < \dots < d_r$ are the positions of the descent tops. Then $w_{d_1} = A$ as noted above and $w_{d_2} > \dots > w_{d_r} > w_n$ for else two of these entries would form the 1, 2 of a 102 pattern.

Now consider an illustrative example with $n = 23$, $m = 14$, $r = 4$, $\{d_1, d_2, d_3, d_4\} = \{13, 17, 18, 21\}$, $S = \{1, 2, 4, 5, 7, 8, 9, 11, 12\}$ and $A = 12$.

position i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
w_i	0	0	0	0	0	0	0	0	0	0	0	11	12	0	5	8	9	7	0	0	4	1	2	
position i													13					17	18				21	23
skeleton	$S = \{1, 2, 4, 5, 7, 8, 9, 11, 12\}$												12					9	7				4	2

FIGURE 1. A (011,102)-avoiding weak ascent sequence w and its skeleton

Lemma 3.2. *For $w \in \mathcal{A}_n(011, 102)$, knowledge of the positive-support S , the descent locations $d_1, d_2, \dots, d_r$, and $w_{d_1}, w_{d_2}, \dots, w_{d_r}, w_n$ is sufficient to recover w .*

Proof. By assumption, the entries $w_{d_1}, w_{d_2}, \dots, w_{d_r}, w_n$ are already assigned, where $w_{d_1} = \max(S) = A$ and w_n may be 0. From the pool of unassigned entries,

$$\{0^m\} \cup S \setminus \{w_{d_1}, w_{d_2}, \dots, w_{d_r}, w_n\},$$

fill in the blank slots in w as follows. In the rightmost blank slot, place the largest pool entry that is $\leq$ its neighbor on the right. Then working right to left, fill each blank spot with the largest unused pool entry that is $\leq$ its neighbor on the right (there always will be such an entry). This procedure is necessary and sufficient to avoid a 102 pattern. The result obviously avoids 011. The entry $A = w_{d_1}$ already meets the weak ascent condition. So do all later entries because they are smaller than A , and so do all earlier entries because they will form a run of 0s followed by an increasing sequence of entries less than A . $\square$

To count $\mathcal{A}_{n,m,r}(011, 102) := \{w \in \mathcal{A}_n(011, 102) \mid w \text{ contains } m \text{ zeros and has } r \text{ descents}\}$, it suffices, in view of Lemma 3.2, to count “skeletons” as illustrated in Figure 1. With $A = n - m + j$ for some $j \in \{0, 1, \dots, m - 1 - r\}$ as shown above, the descent tops form an r -element subset of $\{n - m + j - 1, \dots, n - 1\}$, so $\binom{m-j-1}{r}$ choices. The positive-support set S is an $(n - m)$ -element subset of $\{1, \dots, n - m + j\}$ that includes $A = n - m + j$, so $\binom{n-m+j-1}{n-m-1}$ choices. Lastly, $w_{d_2}, \dots, w_{d_r}, w_n$ is an r -element subset of $\{0\} \cup (S \setminus \{A\})$, where $|S| = n - m$, so $\binom{n-m}{r}$ choices. Consequently,

$$\begin{aligned} |\mathcal{A}_{n,m,r}(011, 102)| &= \binom{n-m}{r} \sum_{j=0}^{m-1-r} \binom{n-m+j-1}{n-m-1} \binom{m-j-1}{r} \\ &= \binom{n-m}{r} \binom{n-1}{m-1-r}, \end{aligned}$$

which implies

$$\begin{aligned} |A_n(011, 102)| &= \sum_{m=1}^n \sum_{r=0}^{\min(n-m, m-1)} \binom{n-m}{r} \binom{n-1}{m-1-r} \\ &= \sum_{m=1}^n \binom{2n-m-1}{m-1} = F_{2n-1}. \end{aligned}$$

3.5. Class 21. Case {010,021} Let $A_m(x)$ (respectively, $B_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(010, 021)$ with a root a_m (respectively, $b_{m,j}$). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\ B_{m,j}(x) &= x + x \sum_{i=j}^{m+1} B_{m+1,i}(x). \end{aligned}$$

Define $A(x; v) = \sum_{m \geq 1} A_m(x) v^{m-1}$ and $B(x; v, u) = \sum_{m \geq 1} \sum_{j=1}^m B_{m,j}(x) v^{m-1} u^{m-j}$. Then the above recurrences can be written as

$$A(x; v) = \frac{x}{1-v} + \frac{x}{v} (A(x; v) - A(x; 0)) + xB(x; v, 1),$$

$$B(x; v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-u)}(B(x; v, u) - uB(x; vu, 1)) - \frac{x}{vu}B(x; v, 0).$$

It is not hard to see that the solution of this system is given by

$$A(x; v) = \frac{(1-v)(1-\sqrt{1-4x})-2x}{2(v^2-v+x)},$$

$$B(x; v, u) = \frac{(1-vu)(1-\sqrt{1-4x})-2x}{2(u^2v^2-uv+x)(1-v)}.$$

Hence, $A(x; 0) = \frac{1-\sqrt{1-4x}}{2x} - 1$.

Case {011,201} The aim here is to prove

$$|\mathcal{A}_n(011, 201)| = C_n, \text{ the } n\text{th Catalan number.}$$

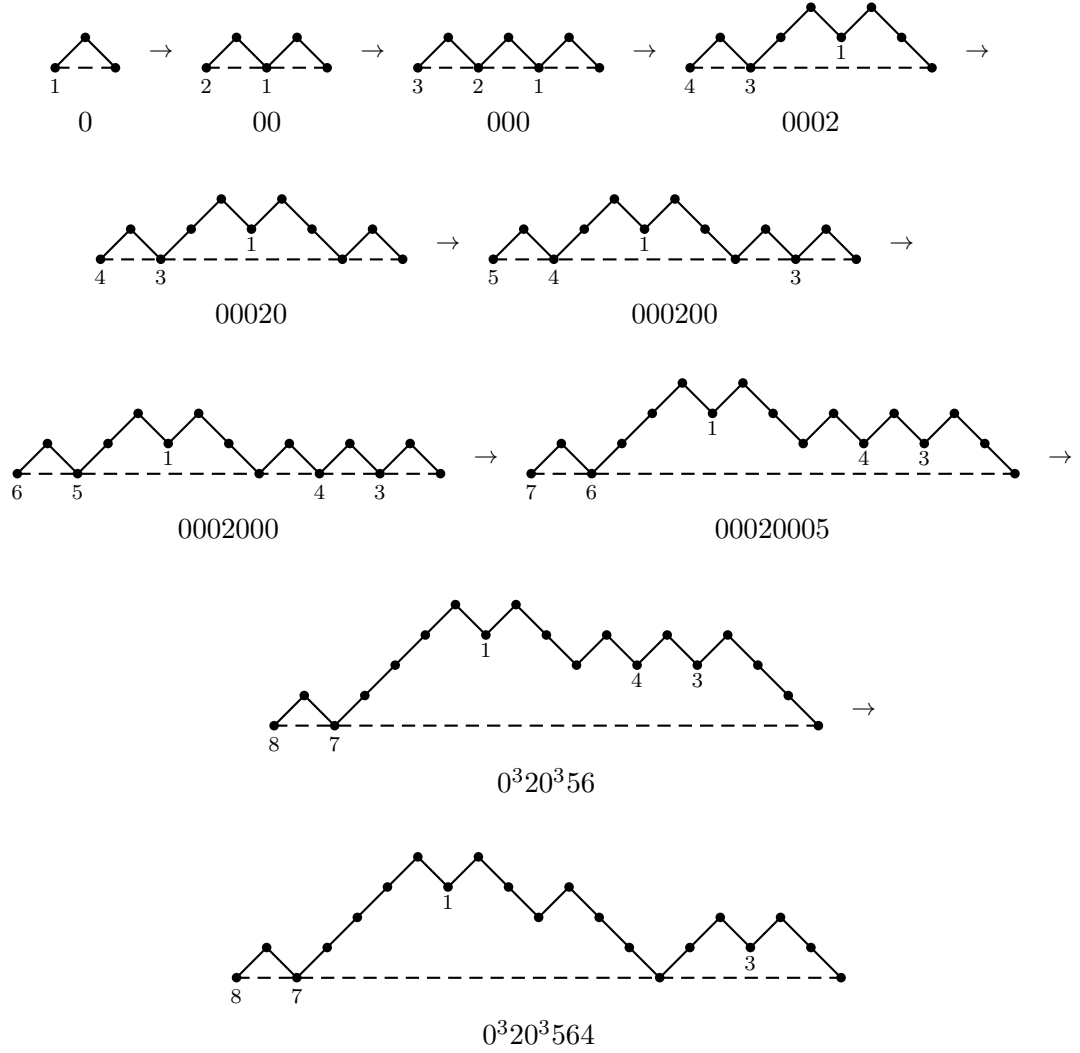
Weak ascent sequences avoiding 011 and 201 can be characterized as weak ascent sequences in which all the nonzero entries are distinct, to avoid 011, and after each descent $b > a$, no later entry lies in the interval $[a+1, b-1]$, to avoid 201.

We will define a bijection ϕ from $\mathcal{A}_n(011, 201)$ to Dyck n -paths, well known to be counted by C_n . First we recall some nomenclature about Dyck paths. A Dyck n -path is a lattice path of n upsteps $U = (1, 1)$ and n downsteps $D = (1, -1)$ that stays weakly above the horizontal line, called *ground level*, that joins its initial and terminal points (vertices). We have the usual notions of size (number of upsteps), ascent (maximal sequence of contiguous upsteps), descent, height (above ground level), valley (contiguous DU), valley vertex (the vertex between the D and U), and return downstep (that returns the path to ground level). A Dyck path with exactly one return downstep is *indecomposable* and every nonempty Dyck path decomposes uniquely (at its returns) into a concatenation of indecomposable Dyck subpaths, its *components*. To *elevate* a Dyck path means to prepend an upstep and append a downstep, thereby producing an indecomposable Dyck path, sometimes called an elevated Dyck path. The *terminal descent* is the maximal string of contiguous downsteps that ends the path.

The *matching* downstep of an upstep U_0 in a Dyck path is the terminal downstep of the shortest Dyck subpath that starts at U_0 . Note that upsteps/downsteps come in matching pairs. Also, the *long-Dyck matching* downstep is the terminal downstep of the longest Dyck subpath that starts at U_0 . Of course, more than one upstep may have the same long-Dyck matching downstep.

A UDU incident with ground level is called a *low UDU*, otherwise a *high UDU*. An upstep whose long-Dyck matching downstep is on the terminal descent is said to *reach* the terminal descent. Note that all upsteps in the last component extending from height 1 to height 2 reach the terminal descent. These notions are illustrated in Figure X below.

The bijection ϕ sends noninitial 0s to valleys, descents to 0 to $DDUD$ s, and descents to a nonzero integer to DDU U's. In particular, the total number of descents in the avoider equals the number of DDU U's in the Dyck path. Also, ϕ sends trailing 0s to trailing UD s; in particular, an avoider ends with a positive integer if and only if the corresponding Dyck path ends DD . For an avoider $w \in \mathcal{A}_n(011, 201)$, the construction of the Dyck path $P = \phi(w)$ is iterative and produces a label on the initial vertex and on the valley vertex of each UDU in P in such a way that all labels are distinct, say the labels form a set L , and with r denoting the number of descents in w , the interval $[0, r+1]$ is the disjoint union of L and the distinct



The avoider $0^3 20^3 564$ has maximum entry $m = 6$ and two eligible-to-append integers that exceed 6, namely 7, 8, and one that falls below 6, namely 3. The former label the origin and low *UDUs*. The latter label high *UDUs* that reach the terminal descent.

FIGURE 3. Construction of bijection ϕ

Proof. The proof is by induction. We consider the various cases for appending an eligible integer to an avoider.

The result holds in case $a = 0 < w_n$ because there are no changes in label, the maximum entry “ m ” remains unchanged, all currently eligible positive integers below m lose their eligibility and all high *UDUs* fail to reach the terminal descent due to the newly introduced terminal *UD*. In case $a = 0 = w_n$, the only change in eligibility is the addition of $r + 2$ (due to the additional weak ascent) and the construction produces labels $m + 1, \dots, r + 1, r + 2$ at ground level, while eligibility of integers below m and labels above ground level are unchanged.

In case $a = r + 1$, the integers $r, r - 1, \dots, m + 1$ label the low $UDUs$ by the induction hypothesis. They remain eligible but are now less than the new maximum. Their $UDUs$ get raised one unit, becoming high $UDUs$ that reach the terminal descent because they are now in the last (and only) component between heights 1 and 2. Also, eligibility of integers below m remains unchanged as do the high UDU upsteps whose long-Dyck path reaches the terminal descent.

In case $m < a \leq r$, the low $UDUs$ labelled $a - 1, a - 2, \dots, m + 1$ get raised one unit, thereby becoming high $UDUs$ and, as in the previous case, they all reach the terminal descent. At the same time, these integers lose their eligibility. The new maximum letter in the avoider is a and $a + 1, a + 2, \dots, r + 1, r + 2$ are the new labels on the low $UDUs$ and the initial vertex. Other labels and eligibility are unchanged and so all is well.

Lastly, for the case $0 < a < m$, where a labels a high UDU , currently eligible positive integers between a and m lose their eligibility while those below a remain eligible. But the latter integers occur *after* vertex a and get transferred to a new last component where they still reach the terminal descent, and the former integers label $UDUs$ that are no longer in the last component and so certainly don't reach the terminal descent. $\square$

Due to Lemma 3.3, the procedure works to map $\mathcal{A}_n(011, 201)$ to Dyck n -paths. To see that the map can be reversed, note that the 5 cases considered above to append an eligible integer to an existing avoider can be distinguished as follows. Case 3) occurs when the Dyck path has only one component. Otherwise, let C_{-2} denote the second last component and C_{-1} the last component. In case 1) C_{-2} ends DD and $C_{-1} = UD$, and in case 2) $C_{-2} = C_{-1} = UD$. This covers all cases in which $C_{-1} = UD$. Case 4) elevates a segment following a UD at ground level, so $C_{-2} = UD$ and C_{-1} starts UU ; in case 5) C_{-2} ends DD and C_{-1} starts UU . Thus from $\phi(w_1 \dots w_{n-1} w_n)$, we can retrieve $\phi(w_1 \dots w_{n-1})$ and, by induction, $w_1 \dots w_{n-1}$. The eligible integers to extend $w_1 \dots w_{n-1}$ can then be ascertained and, according to Lemma 3.3, used to label all of the low $UDUs$ and those of the high $UDUs$ that reach the terminal descent in $\phi(w_1 \dots w_{n-1})$. Then the actual eligible integer w_n can readily be found from the action that took $\phi(w_1 \dots w_{n-1})$ to $\phi(w_1 \dots w_{n-1} w_n)$.

4. AVOIDING A TRIPLE OF PATTERNS OF LENGTH 3

By Table 4 there are exactly 110 trivial and 40 nontrivial candidate classes. The aim of this section is to show that the 40 nontrivial candidate classes of triples of length 3 patterns are indeed 40 A -Wilf equivalences. Table 3 describes all these 40 nontrivial candidate classes including the corresponding succession rules of the generating trees and explicit formulas (if they exist) for the generating function. The classes 47, 63, 95, 105, 114, 140 are treated using bijective methods. The four open cases, classes 120, 123, 132, and 139, are in blue and indicated by the double question marks.

Table 2: Rules of generating trees for weak ascent sequences avoiding a triple of length-3 patterns.

Beginning of Table 2			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
2	$\{000, 001, 010\}$	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \dots m$, $b_m = a_m m$	
	$\{000, 001, 011\}$	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \dots m$, $b_m = a_m 0$	
	$\{001, 010, 011\}$	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 0^m$, $b_m = 01 \dots m$	
	$\{001, 010, 012\}$ $\{001, 011, 012\}$	$a_1 \rightsquigarrow a_2, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 0^m$	$\frac{x(1+x)}{1-x}$

Continuation of Table 2			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
5	$\{001,011,100\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0^m, b_m = 01 \cdots m, c_m = b_m 0$	$\frac{x(1+x+x^2)}{1-x}$
	$\{001,011,120\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_1 \rightsquigarrow a_2, b_2; b_m \rightsquigarrow b_{m+1}$, where $a_m = 0^m, b_m = 01 \cdots m$	
	$\{001,012,100\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0^m, b_m = 01^m, c_m = b_m 0$	
	$\{001,012,110\}$	$a_1 \rightsquigarrow a_2, 01; a_m \rightsquigarrow a_{m+1}; 01 \rightsquigarrow a_2, a_3$, where $a_m = 0^m$	
6	$\{000,001,021\}$	$0 \rightsquigarrow 00, a_1; a_m \rightsquigarrow b_{m-1}, c_m, a_{m+1}; c_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m, c_m = 01 \cdots mm, b_m = c_m 0$	$\frac{x(1+x+x^2+x^3)}{1-x}$
	$\{000,001,120\}$	$0 \rightsquigarrow 00, a_1; a_m \rightsquigarrow b_{m-1}, c_m, a_{m+1}; c_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m, c_m = 01 \cdots mm, b_m = c_m(m-1)$	
7	$\{000,001,110\}$	$0 \rightsquigarrow 00, a_1; a_m \rightsquigarrow (b_{m-1})^m, b_m, a_{m+1}$, where $a_m = 01 \cdots m, b_m = a_m m$	$\frac{x}{(1-x)^2}$
	$\{001,010,021\}$ $\{001,010,100\}$ $\{001,010,101\}$ $\{001,010,102\}$ $\{001,010,110\}$ $\{001,010,120\}$ $\{001,010,201\}$ $\{001,010,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+2}, b_{m+1}$, where $a_m = 0^m, b_m = 01 \cdots m$	
	$\{001,011,021\}$ $\{001,011,101\}$ $\{001,011,102\}$ $\{001,011,110\}$ $\{001,011,201\}$ $\{001,011,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m, b_m = 01 \cdots m$	
	$\{001,012,021\}$ $\{001,012,101\}$ $\{001,012,102\}$ $\{001,012,120\}$ $\{001,012,201\}$ $\{001,012,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m, b_m = 01^m$	
	$\{000,001,100\}$ $\{000,001,101\}$ $\{000,001,102\}$ $\{000,001,201\}$	$0 \rightsquigarrow 00, a_1;$ $a_m \rightsquigarrow b_{m-1,0}, \dots, b_{m-1,m-2}, c_{m-1}, c_m, a_{m+1};$ $b_{m,j} \rightsquigarrow b_{m,0}, \dots, b_{m,j-1}; c_m \rightsquigarrow b_{m,0}, \dots, b_{m,m-1},$ where $a_m = 01 \cdots m, b_{m,j} = a_m m j, c_m = a_m m$	
	$\{010,011,012\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}; b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j-1}$	
10	$\{000,001,100\}$ $\{000,001,101\}$ $\{000,001,102\}$ $\{000,001,201\}$	$0 \rightsquigarrow 00, a_1;$ $a_m \rightsquigarrow b_{m-1,0}, \dots, b_{m-1,m-2}, c_{m-1}, c_m, a_{m+1};$ $b_{m,j} \rightsquigarrow b_{m,0}, \dots, b_{m,j-1}; c_m \rightsquigarrow b_{m,0}, \dots, b_{m,m-1},$ where $a_m = 01 \cdots m, b_{m,j} = a_m m j, c_m = a_m m$	$\frac{x(1+x)}{1-x-x^2}$
	$\{010,011,012\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}; b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j-1}$	
21	$\{000,012,100\}$	$0 \rightsquigarrow 00, 01; 00 \rightsquigarrow 001, 002; 01 \rightsquigarrow 010, 001; 001 \rightsquigarrow 0011;$ $002 \rightsquigarrow 0021, 0022; 010 \rightsquigarrow 0021; 0022 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 5x^4 + 2x^6 + x^7$
	$\{000,012,102\}$ $\{000,012,120\}$ $\{000,012,201\}$ $\{000,012,210\}$	$0 \rightsquigarrow 00, 01; 00 \rightsquigarrow 001, 002; 01 \rightsquigarrow 010, 001; 001 \rightsquigarrow 0011;$ $002 \rightsquigarrow 001, 0022; 010 \rightsquigarrow 0101; 0022 \rightsquigarrow 00221;$ $00221 \rightsquigarrow 002211$	
22	$\{000,011,021\}$	$a_m \rightsquigarrow b_m, a_{m+1}; b_0 \rightsquigarrow d_1, c_2; b_m \rightsquigarrow c_{m+1}; c_m \rightsquigarrow c_{m+1};$ $d_m \rightsquigarrow d_{m+1}, c_{m+2}$, where $a_m = 01 \cdots m, b_m = a_m 0,$ $c_m = 0a_{m-2}m, d_m = 0a_m$	$\frac{x(1+x^2)}{(1-x)^2}$
	$\{000,011,102\}$	$0 \rightsquigarrow 00, 01; 00 \rightsquigarrow a_1, b_2; 01 \rightsquigarrow 010, b_2;$ $a_m \rightsquigarrow a_{m+1}, b_{m+2}; b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0012 \cdots m, b_m = a_{m-2}m, c_m = b_m(m-1)$	
	$\{000,011,120\}$	$0 \rightsquigarrow a_0, 01; 01 \rightsquigarrow 010, 012; 010 \rightsquigarrow 012; 012 \rightsquigarrow d_3;$ $a_m \rightsquigarrow a_{m+1}, b_{m+2}; b_m \rightsquigarrow c_m, d_{m+1}; c_m \rightsquigarrow d_{m+1};$ $d_m \rightsquigarrow d_{m+1}$, where $a_m = 001 \cdots m, b_m = a_{m-2}m,$ $c_m = b_m(m-1), d_m = a_{m-3}(m-1)m$	
	$\{001,021,100\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_1 \rightsquigarrow c_1, b_2, d_2;$ $b_m \rightsquigarrow c_m, b_{m+1}; d_m \rightsquigarrow c_m, b_{m+1}, d_{m+1}$, where $a_m = 0^m, b_m = 01^m, c_m = b_m 0, d_m = 01 \cdots m$	
	$\{001,021,110\}$ $\{001,110,120\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+1}, a_{m+2}, b_{m+1},$ where $a_m = 0^m, b_m = 01 \cdots m$	
	$\{001,021,120\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_1 \rightsquigarrow a_2, b_2, c_2;$ $b_m \rightsquigarrow a_{m+1}, b_{m+1}; c_m \rightsquigarrow a_{m+2}, c_{m+1}$, where $a_m = 0^m,$ $b_m = 01^m, c_m = 01 \cdots m$	
	$\{001,100,110\}$	$a_1 \rightsquigarrow a_2, b_1; b_m \rightsquigarrow (c_m)^m, a_{m+2}, b_{m+1}$, where $a_m = 0^m,$ $b_m = 01 \cdots m, c_m = b_m 0$	
	$\{001,100,120\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow c_m, d_{m+1}, b_{m+1};$ $d_m \rightsquigarrow c_m, d_{m+1}$, where $a_m = 0^m, b_m = 01 \cdots m,$ $c_m = 01^m 0, d_m = 01^m$	
23	$\{000,011,100\}$ $\{000,011,101\}$		

Continuation of Table 2			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
	$\{000,011,110\}$ $\{000,011,201\}$ $\{000,011,210\}$	$0 \rightsquigarrow a_0, 01; 01 \rightsquigarrow 010; 010 \rightsquigarrow c_2; a_m \rightsquigarrow a_{m+1}, b_{m+2};$ $b_m \rightsquigarrow c_m, b_{m+1}; c_m \rightsquigarrow c_{m+1}, \text{ where } a_m = 001 \cdots m,$ $b_m = a_{m-2}m, c_m = b_m(m-1)$	
	$\{001,021,101\}$ $\{001,021,102\}$ $\{001,021,201\}$ $\{001,021,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+1}, c_{m+1}, b_{m+1};$ $c_m \rightsquigarrow a_{m+1}, c_{m+1}, \text{ where } a_m = 0^m, b_m = 01 \cdots m,$ $c_m = 01^m$	
	$\{001,100,210\}$	$a_1 \rightsquigarrow a_2, d_1; a_m \rightsquigarrow a_{m+1}; d_m \rightsquigarrow (c_m)^m, b_{2,m}, d_{m+1};$ $b_{i,j} \rightsquigarrow (c_{i+j-1})^j, b_{i+1,j}, \text{ where } a_m = 0^m,$ $b_{i,j} = 01 \cdots (j-1)j^i, c_m = 01^m 0, d_m = 01 \cdots m$	
	$\{001,101,110\}$ $\{001,102,110\}$ $\{001,110,201\}$ $\{001,110,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow (a_{m+1})^m, a_{m+2}, b_{m+1},$ $\text{ where } a_m = 0^m, b_m = 01 \cdots m$	
	$\{001,101,120\}$ $\{001,102,120\}$ $\{001,120,201\}$ $\{001,120,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+1}, c_{m+1}, b_{m+1};$ $c_m \rightsquigarrow a_{m+1}, c_{m+1}, \text{ where } a_m = 0^m, b_m = 01 \cdots m,$ $c_m = 01^m$	
	$\{010,012,021\}$ $\{011,012,021\}$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow b_{m+1}, \text{ where } a_m = 0^m,$ $b_m = a_m 1$	$\frac{x(1-x+x^2)}{(1-x)^3}$
24	$\{001,100,101\}$ $\{001,100,102\}$ $\{001,100,201\}$	$a_1 \rightsquigarrow a_2, d_1; a_m \rightsquigarrow a_{m+1};$ $d_m \rightsquigarrow c_{m,1}, \dots, c_{1,m}, b_{2,m}, d_{m+1};$ $b_{i,j} \rightsquigarrow c_{i+j-1,1}, \dots, c_{i,j}, b_{i+1,j};$ $c_{i,j} \rightsquigarrow c_{i+j-1,1}, \dots, c_{i+1,j-1}, \text{ where } a_m = 0^m,$ $d_m = 01 \cdots m, b_{i,j} = 01 \cdots (j-1)j^i,$ $c_{i,j} = 01 \cdots (j-1)j^i(j-1)$	$\frac{x}{(1-x)(1-x-x^2)}$
26	$\{001,101,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow a_{m+2}, c_{m+1}, b_{m+1};$ $c_m \rightsquigarrow a_{m+1}, c_{m+1}, \text{ where } a_m = 0^m, b_m = 01 \cdots m,$ $c_m = 01^m$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
	$\{001,102,210\}$ $\{001,201,210\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1}; b_m \rightsquigarrow (a_{m+1})^m, c_{2,m}, b_{m+1};$ $c_{i,j} \rightsquigarrow (a_{i+j})^j, c_{i+1,j}, \text{ where } a_m = 0^m, b_m = 01 \cdots m,$ $c_{i,j} = 01 \cdots (j-1)j^i$	
	$\{011,012,210\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}; b_{m,j} \rightsquigarrow b_{m,j}, (c_m)^{j-1},$ $\text{ where } a_m = 0^m, b_m = a_m j, c_m = a_m 21$	
27	$\{010,012,100\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow c_{m,2}, \dots, c_{m,j-1}, b_{m+1,j};$ $c_{m,j} \rightsquigarrow c_{m,2}, \dots, c_{m,j-1}, \text{ where } a_m = 0^m, b_m = a_m j,$ $c_{m,j} = a_m j(j-1)$	$\frac{x(1-x+x^3)}{(1-x)^2(1-x-x^2)}$
	$\{010,012,110\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j-1}, b_{m+1,j}, \text{ where } a_m = 0^m,$ $b_m = a_m j$	
	$\{011,012,201\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, b_{m,1}, \dots, b_{m,j-1}, \text{ where } a_m = 0^m,$ $b_m = a_m j$	
29	$\{001,101,102\}$ $\{001,101,201\}$ $\{001,102,201\}$	$a_1 \rightsquigarrow a_2, b_1; a_m \rightsquigarrow a_{m+1};$ $b_m \rightsquigarrow a_{m+1}, c_{m,1}, \dots, c_{2,m-1}, c_{2,m}, b_{m+1};$ $c_{i,j} \rightsquigarrow a_{i+j}, c_{i+j-1,1}, \dots, c_{i+1,j-1}, c_{i+1,j}, \text{ where}$ $a_m = 0^m, b_m = 01 \cdots m, c_{i,j} = 01 \cdots (j-1)j^i$	
	$\{010,011,021\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m+1,j+1}, \dots, b_{m+1,m+1}, \text{ where } a_m = 0^m,$ $b_{m,j} = a_m j$	
	$\{010,012,101\}$ $\{010,012,102\}$ $\{010,012,120\}$ $\{010,012,201\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j-1}, b_{m+1,j}, \text{ where } a_m = 0^m,$ $b_{m,j} = a_m j$	
	$\{011,012,101\}$ $\{011,012,102\}$ $\{011,012,110\}$ $\{011,012,120\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}; b_{m,j} \rightsquigarrow b_{m,1}, \dots, b_{m,j},$ $\text{ where } a_m = 0^m, b_{m,j} = a_m j$	
35	$\{010,011,100\}$		$\frac{x}{1-2x}$

Continuation of Table 2			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
	$\{010,011,101\}$	$\stackrel{a}{\sim} \{010,011\}$	
36	$\{010,011,110\}$		
	$\{012,021,100\}$ $\{012,021,101\}$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow b_{m+1}, c_m, c_m \rightsquigarrow c_{m+1},$ where $a_m = 0^m, b_m = a_m 1, c_m = b_m 0$	$\frac{x(1-2x+3x^2-x^3)}{(1-x)^4}$
	$\{012,021,110\}$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow b_{m+1}, c_m, c_m \rightsquigarrow c_{m+1},$ where $a_m = 0^m, b_m = a_m 1, c_m = b_m 1$	
41	$\{012,100,210\}$ $\{012,110,210\}$	Section 4.1	Section 4.1
42	$\{012,100,201\}$ $\{012,110,201\}$	Section 4.2	Section 4.2
43	$\{012,100,102\}$ $\{012,100,120\}$	$\stackrel{a}{\sim} \{012,100\}$	
45	$\{011,021,102\}$ $\{012,101,201\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow c_m, b_{m,1}, \dots, b_{m,j-1}, b_{m+1,j}; c_m \rightsquigarrow c_{m+1},$ where $a_m = 0^m, b_{m,j} = a_m j, c_m = a_m 10$	$\frac{x(2x^2-2x+1)}{(1-x)^2(1-2x)}$
	$\{012,021,102\}$ $\{012,021,120\}$ $\{012,021,201\}$		
	$\{012,021,210\}$ $\{012,102,110\}$	$\stackrel{a}{\sim} \{012,021\}$	
	$\{012,110,120\}$	$\stackrel{a}{\sim} \{012,110\}$	
	$\{011,021,100\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow c_{m,j}, b_{m+1,j+1}, \dots, b_{m+1,m+1};$ $c_{m,j} \rightsquigarrow c_{m+1,j+1}, \dots, c_{m+1,m+1},$ where $a_m = 0^m,$ $b_{m,j} = a_m j, c_{m,j} = b_{m,j} 0$	
47	$\{011,021,120\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow c_{m,j}, d_{m,j+1}, \dots, d_{m,m+1};$ $c_{m,j} \rightsquigarrow c_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1};$ $d_{m,j} \rightsquigarrow d_{m+1,j+1}, \dots, d_{m+1,m+2},$ where $a_m = 0^m,$ $b_{m,j} = a_m j, c_{m,j} = b_{m,j} 0, d_{m,j} = a_m 1j$	Section 4.3
	$\{011,102,120\}$		
48	$\{012,101,102\}$ $\{012,101,120\}$	$\stackrel{a}{\sim} \{012,101\}$	$\frac{x(1-x+x^2)}{1-3x+2x^2-x^3}$
55	$\{012,102,201\}$ $\{012,120,201\}$	$\stackrel{a}{\sim} \{012,201\}$	Section 3.3
	$\{012,102,210\}$ $\{012,120,210\}$	$\stackrel{a}{\sim} \{012,210\}$	
58	$\{000,021,100\}$ $\{000,021,201\}$	$\stackrel{a}{\sim} \{000,021\}$	Class 15 in Table 1 see [2], $\frac{x(1-x)}{1-3x+x^2}$
	$\{000,021,210\}$		
60	$\{011,101,102\}$ $\{011,102,110\}$	$\stackrel{a}{\sim} \{011,102\}$	See Section 4.4
	$\{012,102,120\}$	$\stackrel{a}{\sim} \{012\}$	
63	$\{011,021,101\}$ $\{011,021,110\}$ $\{011,021,201\}$	$\stackrel{a}{\sim} \{011,021\}$	See Section 4.4
	$\{011,021,210\}$ $\{011,100,201\}$		
64	$\{011,101,120\}$ $\{011,110,120\}$	$\stackrel{a}{\sim} \{011,120\}$	
68	$\{011,100,101\}$ $\{011,100,110\}$	$\stackrel{a}{\sim} \{011,100\}$	
78	$\{010,021,100\}$ $\{010,021,101\}$ $\{010,021,102\}$ $\{010,021,110\}$ $\{010,021,120\}$ $\{010,021,201\}$		$\frac{1-\sqrt{1-4x}}{2x} - 1$; Class 21 in Table 1
	$\{010,021,210\}$	$\stackrel{a}{\sim} \{010,021\}$	
	$\{011,101,201\}$ $\{011,110,201\}$	$\stackrel{a}{\sim} \{011,201\}$	
80	$\{011,101,210\}$ $\{011,110,210\}$	$\stackrel{a}{\sim} \{011,210\}$	
95	$\{010,101,120\}$ $\{010,110,120\}$		Section 4.5
105	$\{010,100,201\}$ $\{010,101,201\}$		Section 4.6
114	$\{021,100,120\}$ $\{021,101,120\}$ $\{021,110,120\}$		Section 4.7
117	$\{021,102,201\}$		

Continuation of Table 2			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$ /Reference
	$\{021,102,210\}$	$\overset{a}{\sim} \{021, 102\}$	
119	$\{021,110,201\}$		
	$\{021,110,210\}$	$\overset{a}{\sim} \{021, 110\}$	
120	$\{021,120,201\}$		
	$\{021,120,210\}$	$\overset{a}{\sim} \{021, 120\}$	
	$\{102,110,120\}$	??	
123	$\{021,100,201\}$		
	$\{021,100,210\}$	$\overset{a}{\sim} \{021, 100\}$	
	$\{101,102,110\}$	??	
124	$\{021,101,201\}$		
	$\{021,101,210\}$	$\overset{a}{\sim} \{021, 101\}$	
132	$\{102,110,210\}$	??	
	$\{102,120,210\}$		
139	$\{021,201,210\}$	$\overset{a}{\sim} \{021\}$	
	$\{100,110,201\}$	??	
	$\{101,110,201\}$	??	
140	$\{100,120,201\}$		Section 4.8
	$\{110,120,201\}$		
150	$\{100,201,210\}$ $\{101,201,210\}$	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow (c_{m,j})^j, b_{m+1,j}, \dots, b_{m+1,m+1};$ $c_{m,j} \rightsquigarrow c_{m+1,j}, d_{m,j,j+1}, \dots, d_{m,j,m+1};$ $d_{m,j,i} \rightsquigarrow (c_{m+1,i})^{j-i}, d_{m+1,j,i}, \dots, d_{m+1,j,m+2},$ where $a_m = 0^m, b_{m,j} = a_m j, c_{m,j} = a_m j 0, d_{m,j,i} = a_m j 0 i$	
End of Table 2			

4.1. Class 41. Case $\{012, 100, 210\}$ By our algorithm, we obtain that the tree $\mathcal{T}(012, 100, 210)$ with a root a_1 has the following rules

$$\begin{aligned}
 a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\
 b_{m,j} &\rightsquigarrow c_{m,j}, (d_m)^{j-1}, b_{m+1,j}, \\
 c_{m,j} &\rightsquigarrow (d_{m+1})^{j-1}, c_{m+1,j},
 \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j 0$, and $d_m = a_m 21$.

Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 100, 210)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, d_m). Thus,

$$\begin{aligned}
 A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\
 B_{m,j}(x) &= x + xC_{m,j}(x) + (j-1)x D_m(x) + xB_{m+1,j}(x), \\
 C_{m,j}(x) &= x + (j-1)x D_{m+1}(x) + xC_{m+1,j}(x), \\
 D_m(x) &= x.
 \end{aligned}$$

By induction on m, j , we have that $D_m(x) = x$, $C_{m,j}(x) = \frac{x(1+(j-1)x)}{1-x}$, and $B_{m,j}(x) = \frac{x(1+(j-1)x)}{(1-x)^2}$. Hence,

$$A_m(x) = x + xA_{m+1}(x) + \sum_{j=1}^m \frac{x(1+(j-1)x)}{(1-x)^2},$$

which implies that

$$A_1(x) = \frac{x(2-2x+x^2)(1-3x+4x^2-x^3)}{(1-x)^5}.$$

Case {012,110,210} By our algorithm, we obtain that the tree $\mathcal{T}(012, 110, 210)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow b_{m,j}, (c_m)^{j-1}, c_{m+1}, \\ c_m &\rightsquigarrow c_{m+1}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, and $c_m = a_m 11$.

Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 110, 210)$ with a root a_m (respectively, $b_{m,j}$, c_m). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\ B_{m,j}(x) &= x + xB_{m,j}(x) + (j-1)x C_m(x) + xC_{m+1}(x), \\ C_m(x) &= x + xC_{m+1}(x). \end{aligned}$$

By induction on m, j , we have that $C_m(x) = \frac{x}{1-x}$ and $B_{m,j}(x) = \frac{x(1+(j-1)x)}{(1-x)^2}$. Hence,

$$A_m(x) = x + xA_{m+1}(x) + \sum_{j=1}^m \frac{x(1+(j-1)x)}{(1-x)^2},$$

which implies that

$$A_1(x) = \frac{x(2-2x+x^2)(1-3x+4x^2-x^3)}{(1-x)^5}.$$

4.2. Class 42. Case {012,100,201} By our algorithm, we obtain that the tree $\mathcal{T}(012, 100, 201)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_m, d_{m,2}, \dots, d_{m,j}, b_{m+1,j}, \\ c_m &\rightsquigarrow c_{m+1}, \\ d_{m,j} &\rightsquigarrow e_m, d_{m,2}, \dots, d_{m,j-1}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 10$, $d_{m,j} = a_m j(j-1)$, and $e_m = a_m 210$.

Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$, $D_{m,j}(x)$, $E_m(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 100, 201)$ with a root a_m (respectively, $b_{m,j}$, c_m , $d_{m,j}$, e_m). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\ B_{m,j}(x) &= x + xC_m(x) + x \sum_{i=2}^j D_{m,i}(x) + xB_{m+1,j}(x), \\ C_m(x) &= x + xC_{m+1}(x), \end{aligned}$$

$$D_{m,j}(x) = x + xE_m(x) + x \sum_{i=2}^{j-1} D_{m,i}(x),$$

$$E_m(x) = x.$$

By induction on m, j , we have that $E_m(x) = x$, $D_{m,j}(x) = x(1+x)^{(j-1)}$, $C_m(x) = \frac{x}{1-x}$, and $B_{m,j}(x) = \frac{x^3}{(1-x)^2} + \frac{x(1+x)^j}{1-x}$. Hence,

$$A_m(x) = x + xA_{m+1} + x \sum_{j=1}^m \left(\frac{x^3}{(1-x)^2} + \frac{x(1+x)^j}{1-x} \right),$$

which implies that

$$A_1(x) = \frac{x(1-3x+4x^2-x^3-2x^4)}{(1-x)^4(1-x-x^2)}.$$

Case {012,110,201} By our algorithm, we obtain that the tree $\mathcal{T}(012, 110, 201)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow b_{m,1}, d_{m,2}, \dots, d_{m,j}, c_m, \\ c_m &\rightsquigarrow c_{m+1}, \\ d_{m,j} &\rightsquigarrow c_{m-1}, d_{m,2}, \dots, c_m, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 11$, and $d_{m,j} = a_m j(j-1)$.

Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$, $D_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(012, 110, 201)$ with a root a_m (respectively, $b_{m,j}$, c_m , $d_{m,j}$). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\ B_{m,j}(x) &= x + xB_{m,1}(x) + x \sum_{i=2}^j D_{m,i}(x) + xC_m(x), \\ C_m(x) &= x + xC_{m+1}(x), \\ D_{m,j}(x) &= x + xC_{m-1}(x) + x \sum_{i=2}^{j-1} D_{m,i}(x) + xC_m(x). \end{aligned}$$

By induction on m, j , we have that $D_{m,j}(x) = \frac{x(1+x)^{(j-1)}}{1-x}$, $C_m(x) = \frac{x}{1-x}$, and $B_{m,j}(x) = \frac{x^3}{(1-x)^2} + \frac{x(1+x)^j}{1-x}$. Hence,

$$A_m(x) = x + xA_{m+1} + x \sum_{j=1}^m \left(\frac{x^3}{(1-x)^2} + \frac{x(1+x)^j}{1-x} \right),$$

which implies that

$$A_1(x) = \frac{x(1-3x+4x^2-x^3-2x^4)}{(1-x)^4(1-x-x^2)}.$$

4.3. Class 47. Case {011,102,120}:

Lemma 4.1. *Weak ascent sequences avoiding 011, 102 and 120 can be characterized as weak ascent sequences in which (i) all the nonzero entries are distinct, to avoid 011, (ii) the entries greater than the last entry are decreasing, to avoid 120, and (iii) in case the last entry is nonzero, the entries less than the last entry are increasing and occur after all the 0s, in both cases to avoid 102.* $\square$

For example, $w = 0^9 9080012745$ (with the superscript denoting repetition) meets all three conditions in Lemma 4.1 because 9875 is decreasing and 1245 is increasing, and no 0 occurs after an entry smaller than 5.

Lemma 4.2. *Suppose $(w_i)_{i=1}^n$ is a sequence of nonnegative integers that meets all three conditions in Lemma 4.1. Then w is a weak ascent sequence provided only that its largest entry w_j satisfies $w_j \leq j - 1$.*

Proof. Suppose $w_j \leq j - 1$. Then since $w_1 \dots w_j$ must be weakly increasing by condition (iii), $w_1 \dots w_{j-1}$ contains $j - 2$ ascents and so w_j satisfies the weak ascent condition. Entries after w_j , being smaller than w_j , satisfy the weak ascent condition automatically (a fortiori, if you prefer the Latin phrase in its British spelling), and for $\ell < j$, if w_ℓ is positive, then since the entries w_i , $\ell \leq i \leq j$, are increasing by conditions (i) and (iii), we have $w_\ell \leq \ell - 1$ and it too satisfies the weak ascent condition. $\square$

Lemmas 4.1 and 4.2 permit a direct count of $A_n(011, 102, 120)$ by last entry. Fix $k \in [0, n - 2]$ and let us count the sequences $w \in A_n(011, 102, 120)$ with $w_n = k$. Let X denote the decreasing list of entries in w that exceed k and Y denote the increasing list of nonzero entries in w that fall below k . Set $j = |X|$. If $j \geq 1$, then $\max(X) = k + (j - 1) + i$ for some $i \geq 1$, because the smallest possible set comprising X is $\{k + 1, \dots, k + j\}$. For w to satisfy the weak ascent condition, by Lemma 4.2 this maximum, and hence all j elements of X , must occur in position $k + j + i$ or later. So, since the last position in w is occupied by k , there are $\binom{n-k-j-i}{j}$ choices to place the elements of X . Also, X consists of its maximum element $k + j - 1 + i$ and $j - 1$ elements chosen from $k + 1, \dots, k + j - 2 + i$, giving $\binom{j+i-2}{j-1}$ choices for the elements of X .

All told, the number of ways to place entries greater than k is

$$(4.1) \quad 1 \text{ (for case } j = 0) + \sum_{j \geq 1} \sum_{i \geq 1} \binom{j+i-2}{j-1} \binom{n-k-j-i}{j}.$$

The inner sum can be evaluated using identity (5.26) of Table 169 in [6]:

$$\sum_{n-q \leq i \leq \ell-m} \binom{\ell-i}{m} \binom{q+i}{n} = \binom{\ell+q+1}{m+n+1} \quad \text{integers } \ell, m, n \geq 0.$$

This useful identity has a fairly evident combinatorial interpretation: it counts $(m + n + 1)$ -element subsets of $[q + \ell + 1]$ by the value $q + i + 1$ of the $(n + 1)$ -th smallest element of the subset. Putting $\ell = n - k - j$, $q = j - 2$, $m = j$, $n = j - 1$ in Knuth's identity, (4.1) becomes

$$\sum_{j \geq 0} \binom{n-k-1}{2j} = 2^{n-k-2}.$$

Now let $u(n, k) = |\{w \in A_n(011, 102, 120) : w_n = k\}|$, $0 \leq k \leq n-1$. If $k = 0$ and so no entries are smaller than k , the preceding analysis shows $u(n, 0) = 2^{n-2}$. If $k \geq 1$, the entries smaller than k form an arbitrary subset of $[k-1]$, right-justified in w , giving 2^{k-1} choices. So, for $k = n-1$ (with no entries greater than k) we find $u(n, n-1) = 2^{(n-1)-1} = 2^{n-2}$, and for $1 \leq k \leq n-2$, we have $u(n, k) = 2^{k-1} \times 2^{n-k-2} = 2^{n-3}$. So $|A_n(011, 102, 120)| = 2^{n-2}$ [for $k = 0$] + 2^{n-2} [for $k = n-1$] + $\sum_{k=1}^{n-2} 2^{n-3}$, and we obtain the following result.

Theorem 4.3. *For all $n \geq 2$, $|A_n(011, 102, 120)| = (n+2)2^{n-3}$.*

Case {011,021,120}: The following lemma evaluating the sum of the partial sums of row m in Pascal's triangle is easy to establish by reversing the order of summation.

Lemma 4.4. *For all $m \geq 0$, $\sum_{j=0}^m \sum_{i=0}^j \binom{m}{i} = (m+2)2^{m-1}$.*

Lemma 4.5. *Weak ascent sequences avoiding 011, 021 and 120 can be characterized as weak ascent sequences in which (i) the nonzero entries are distinct, to avoid 011, and increasing, to avoid 021, and (ii) there are at most two runs of (contiguous) 0s and, if two runs are present, then the initial run of 0s is separated from the second run by a single nonzero entry, for else a 120 is present.* $\square$

For example, 000200000045101213 of length 15 is a weak ascent sequence, and is in $A_{15}(011, 021, 120)$ because 245101213 is increasing and only one entry separates the two runs of 0s.

It is easy to count the members of $A_n(011, 021, 120)$ with only one run of 0s: if $w \in A_n(011, 021, 120)$ has only one run of 0s, the segment after this run is an arbitrary subset of $[1, n-1]$ in increasing order— 2^{n-1} choices and they all satisfy the weak ascent condition.

In the two-run case, we can count by the separating entry and its position: let $k \in [1, n-2]$ denote the first nonzero entry in w and say it is in position $j \in [k+1, n-1]$ ($j \geq k+1$ to satisfy the weak ascent condition), so that $w_j = k$, $w_{j+1} = 0$, and $w_{j+2} \dots w_n$ consists of a run of zero or more 0s followed by a subset of $\{k+1, \dots, n-2\}$ in increasing order (note that the only loss of a weak ascent in w occurs at $(w_j, w_{j+1}) = (k, 0)$ and so only $n-1$ is ruled out as a possible entry in w). Thus, the $n-j-1$ entries $w_{j+2} \dots w_n$ are determined by a subset of size up to $n-j-1$ chosen from $\{k+1, \dots, n-2\}$ which is a set of size $n-2-k$, yielding

$$\binom{n-2-k}{0} + \binom{n-2-k}{1} + \dots + \binom{n-2-k}{n-j-1}$$

choices for given j, k and they all satisfy the weak ascent condition.

Fix $k \in [1, n-2]$, set $m = n-2-k$, and sum over j to get

$$\begin{aligned} \sum_{j=k+1}^{n-1} \sum_{i=0}^{n-j-1} \binom{n-2-k}{i} &= \sum_{j=0}^{n-2-k} \sum_{i=0}^{n-2-k-j} \binom{n-2-k}{i} \\ &= \sum_{j=0}^m \sum_{i=0}^{m-j} \binom{m}{i} = (m+2)2^{m-1} = (n-k)2^{n-3-k}, \end{aligned}$$

the next to last equality by Lemma 4.4. Consequently,

$$|A_n(011, 021, 120)| = 2^{n-1} [\text{no descents}] + \sum_{k=1}^{n-2} (n-k)2^{n-3-k} [\text{one descent}]$$

$$= (n+2)2^{n-3} \text{ for } n \geq 2,$$

which leads to the following theorem.

Theorem 4.6. *For all $n \geq 2$, $|A_n(011, 021, 120)| = (n+2)2^{n-3}$.*

Case {011,021,100} Weak ascent sequences avoiding 011, 021 and 100 can be characterized as weak ascent sequences in which (i) the nonzero entries are distinct, to avoid 011, and increasing, to avoid 021, and (ii) there are at most two runs of 0s for else a 100 is present and further, if two runs are present, then the second run consists of a single 0 for the same reason. For example, 000000024590101213 of length 15 is a weak ascent sequence, and is in $A_{15}(011, 021, 100)$ because 2459101213 is increasing and the second run of 0s consists of a single 0.

Here is a bijection ϕ from $A_n(011, 021, 100)$ to $A_n(011, 021, 120)$. The bijection ϕ is the identity on the intersection $A_n(011, 021, 100) \cap A_n(011, 021, 120)$. So suppose

$$w \in A_n(011, 021, 100) \setminus A_n(011, 021, 120).$$

Then w has the form $0^r y 0 z$ where $r \geq 1$, y is a factor of length at least 2 (else $w \in A_n(011, 021, 120)$), z is a possibly empty factor, and yz is an increasing sequence of positive integers. Define the *excess* of an increasing sequence of positive integers $u_1 u_2 \dots u_m$ to be $E(u_1 u_2 \dots u_m) := \sum_{i=1}^m (u_{i+1} - u_i - 1)$. Thus $E(u_1 u_2 \dots u_m)$ is a nonnegative integer that measures the deviation of u from a sequence of consecutive integers. Now let $j = r + 1$ so that $a := w_j$ is the first nonzero entry in w , and let $k + 1$ denote the position of the last 0 in w so that $b := w_k$ is the unique descent top in w . Set $e = E(y) = E(w_j w_{j+1} \dots w_k)$. Write y as axb where x may be empty so that $w = 0^r a x b 0 z$. Then

$$\phi(w) = 0^{r-e} a 0^{e+2} x z,$$

of the form $0^{s_1} a 0^{s_2} z'$ and is a weak ascent sequence in $A_n(011, 021, 120)$ that contains a 100 pattern, as we leave the reader to check.

For example, with $w = 0^7 24590101213$, we have $r = 7$, $a = 2$, $b = 9$, $j = 8$, $k = 11$ and $e = (b - a) - (k - j) = 7 - 3 = 4$. So $\phi(w) = 0^3 20^6 45101213$. Note that ϕ preserves the first nonzero entry although not necessarily its position. Also, ϕ discards the one descent top in w and increases the number of 0s by 1.

To invert ϕ , the excess e is evident from the length of the second run of 0s, and then r can be recovered from the length of the first run of 0s. Also, a is the entry bracketed by 0s. Next, if $z' = \epsilon$, then certainly $x = \epsilon$. Otherwise, compute $E(az'_1), E(az'_1 z'_2), \dots$. Then $x = \epsilon$ if $E(az'_1) > e$ and if $E(az'_1) \leq e$, choose the largest i for which $E(az'_1 \dots z'_i) \leq e$. Then $x = z'_1 \dots z'_i$. This determination of x follows from the inequality $e = E(w_1 \dots w_{k-1} w_k) < E(w_1 \dots w_{k-1} w_{k+2})$ which holds because $w_k < w_{k+2}$. Continuing, $j = r + 1$ and $k = i + |x| + 1$. Lastly, $w_k = b$ can be retrieved from the identity $e = b - a - (k - j)$. So ϕ is invertible. $\square$

Hence, we can state the following result.

Theorem 4.7. *For all $n \geq 2$, $|A_n(011, 021, 100)| = (n+2)2^{n-3}$.*

Remark 4.8. *By translating the rules of the $\mathcal{T}(011, 021, 100)$ and the rules $\mathcal{T}(011, 021, 120)$ (see Table 2) to recurrence relations, we obtain that the generating function*

$$G_{\{011,021,100\}}(x) = G_{\{011,021,120\}}(x) = \frac{x(1-x)^2}{(1-2x)^2}.$$

4.4. Class 63. The first 4 triples in this class coincide with (011,021)-avoiders, treated in Section 4.4.2. The fifth triple is treated in Section 4.4.3 below. First, we review *UUDD*-free Dyck paths. See Section 3.5 for a review of Dyck path terminology.

4.4.1. *UUDD*-free Dyck paths. The set of Dyck paths of semilength n that avoid *UUDD* is denoted $\mathcal{B}_n$, counted by sequence A082582 in [7]. For example, $\mathcal{B}_1 = \{UD\}$, $\mathcal{B}_2 = \{UDUD\}$, and $\mathcal{B}_3 = \{UUDD, UDUD\}$. In the rest of this section, we introduce notions regarding *UUDD*-free Dyck paths that will be used in Section 4.4.3. For a *UUDD*-free Dyck path of height 1, the *key D* is the first *D*. In particular, the key downstep of *UD* is its only *D*. Now suppose P is a *UUDD*-free Dyck path of height at least 2. Then P contains a *UU* and the last *UU* in P must be immediately followed by *D* (or it's not the last *UU*), then *U* (or a *UUDD* is present), then *D* (or it's not the last *UU*). So we always have the segment *UDUD* starting with the last *UU*. The *D* immediately following the last *UU* is the *key D* in P , see Figure 4. A downstep is *special* if it is the matching downstep of the middle *U* in a *DUU* pattern, otherwise it is a *normal* downstep. Note that every peak downstep is normal. There is a Dyck subpath, denoted $P(S_0)$, *associated* with a special downstep S_0 : the longest subpath Q such that QU_0RS_0 is a Dyck subpath of P where U_0 is the matching upstep for S_0 . Note that Q and R are both nonempty because S_0 is special. See Figure 4, where the blue lines indicate the matching upstep for each special downstep.

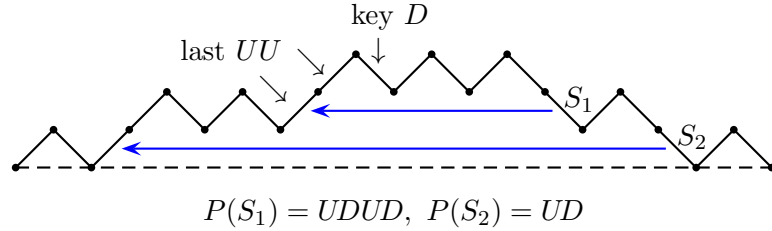


FIGURE 4. A *UUDD*-free Dyck path with 2 special downsteps

Given a *UUDD*-free Dyck path, label all the normal downsteps in P with $0, 1, 2, \dots$ in the order they are visited by the following recursively-defined traversal method. For a path of semilength 1, traversal consists of visiting its only downstep. Otherwise, visit successive *D*s starting at the first *D* after the key *D* until a special *D*, say D_0 is encountered. If and when this happens, traverse $P(D_0)$ (D_0 will remain unlabelled), then visit the key *D* of $P(D_0)$, then continue to the next *D* after D_0 and proceed until the last *D* of P is dealt with. Finally, visit the key *D* of P . See Figure 5 below for a simple example and Figure 7 for further examples.

The *leg* of a downstep in a Dyck path is the longest subpath starting at that downstep and consisting entirely of downsteps. Similarly, the leg of an upstep is the longest subpath ending at that upstep and consisting entirely of upsteps. We say a downstep labelled $m \geq 1$ in a *UUDD*-free Dyck path is *available* if (i) it occurs at or after the key downstep and (ii) the leg L_1 of its matching upstep is longer than its own leg or both legs have the same length and the L_1 leg starts the path. The same criterion holds for the downstep labelled 0 to be available except that the “matching” upstep for downstep 0 is taken to be the first (rather than second) *U* in the *UDUD* segment ending at downstep 0. Note the key downstep is always available

4.4.3. 011,100,201. Introduction. A weak ascent sequence avoids 011, 100, and 201 iff (i) the nonzero entries are distinct, (ii) 0 occurs at most once as a descent bottom, and (iii) after a descent ba , no later entry lies between a and b . For $w \in A_{n-1}(011, 100, 201)$, say a number m is *available* (to extend w) if $wm \in A_n(011, 100, 201)$. For example, with a superscript denoting repetition, if $w = 0^6 36479$, the set of available numbers m is $\mathcal{M} = \{0, 1, 2, 8, 10\}$. Note that 0 is available if and only if all 0s in w occur at the start, and by definition of weak ascent sequence, the maximum available number is $1 + \# \text{ weak ascents}(w)$, which is always available because $\max(w) \leq \# \text{ weak ascents}(w)$. In the next section, we recursively define a bijection $\phi : \mathcal{A}_n(011, 100, 201) \rightarrow \mathcal{B}_{n+1}$. This bijection extends $\phi(w)$ to $\phi(wm)$ in a reversible way for each available number m and relies on the correspondence between available numbers for an avoider and available labels in its corresponding *UDD*-free Dyck path. As will be clear from the construction, ϕ also gives the following correspondences between avoiders w and *UDD*-free Dyck paths $\phi(w)$:

- $\# \{\text{labelled } D\text{s in } \phi(w)\} = 2 + \# \text{ weak ascents}(w)$,
- $\# \{\text{unlabelled } D\text{s in } \phi(w)\} = \# DU\text{Us in } \phi(w) = \# \text{ descents in } w$,
- the labels on $\phi(w)$ comprise the interval $[0, 1 + \# \text{ weak ascents}(w)]$,
- The peak downsteps, being normal, are labelled and the labels consist of 0 on the downstep following the key downstep, $1 + \# \text{ weak ascents}(w)$ on the key downstep, and the positive entries of w on the remaining peak downsteps; in particular, $\# \text{ peaks in } \phi(w) = 2 + \# \text{ positive entries in } w$,
- the last entry in w is the label on the last peak D in $\phi(w)$.

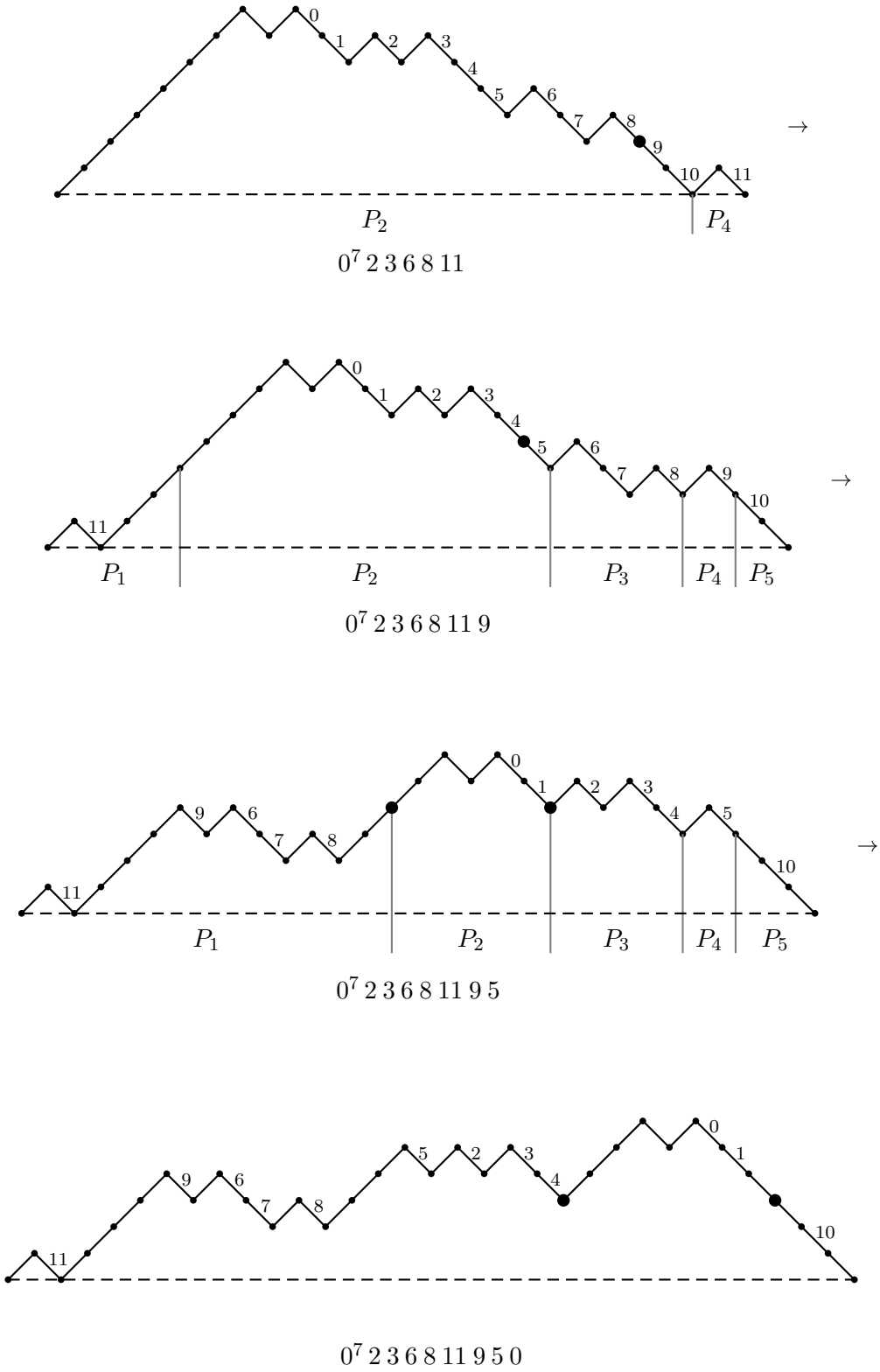
The bijection $\phi : \mathcal{A}_n(011, 100, 201) \rightarrow \mathcal{B}_{n+1}$.

Theorem 4.10. (i) *The map ϕ , presented recursively below, is a bijection from $\mathcal{A}_n(011, 100, 201)$ to $\mathcal{B}_{n+1}$, and (ii) for $n \geq 1$ and $w \in \mathcal{A}_n(011, 100, 201)$ the numbers available to extend w coincide with the available labels for $\phi(w)$.*

Proof. The base case is $\phi(0) = UDUD$ with available numbers 0,1 and available labels also 0,1.

Now suppose $n \geq 2$, $w = (w_i)_{i=1}^n \in \mathcal{A}_n(011, 100, 201)$ and, inductively, $P = \phi((w_i)_{i=1}^{n-1}) \in \mathcal{B}_n$, and the available set $\mathcal{M}$ to append an entry to $(w_i)_{i=1}^{n-1}$ coincides with the labels on the available set of downsteps in P . Set $m = w_n \in \mathcal{M}$. First, suppose m has its maximum possible value, namely, $1 + \# \text{ weak ascents}(w_i)_{i=1}^{n-1}$. In this case, append UD to P to get $\phi(w)$. Next, suppose $0 < m < 1 + \# \text{ weak ascents}(w_i)_{i=1}^{n-1}$. Let j denote the position of the last downstep in the leg of the downstep labelled m . If $j = 2n$, that is, if the j -th step is the last step in P , insert a UD just before the downstep labelled m to get $\phi(w)$. Otherwise, let $k > j$ denote the position of the last U in P and split P into 5 segments P_1, P_2, P_3, P_4, P_5 ending at positions $i-1, j, k-1, k+1, 2n$ respectively. Form P'_2 by inserting a UD just before the downstep labelled m . Then $\phi(w) = P_1 P_4 P_3 P'_2 P_5$ and the inserted UD becomes the last peak in $\phi(w)$. The process in case $m = 0$ is similar except instead of inserting a UD just before the downstep labelled 0 (which would create a *UDD*), the key *UDUD* is elevated, that is, replaced by *UUDUD*.

The construction of ϕ is illustrated for 4 avoiders in Figure 7 below. In the first two paths the heavy dot marks the vertex where UD is inserted in the construction of the next path. Similarly, in the third path where $m = 0$, the two heavy dots enclose the segment of the path that gets elevated in the next path.


 FIGURE 7. Construction of bijection ϕ : avoiders and the corresponding path

A direct description of ϕ^{-1} . Given a UDD -free Dyck path in $\mathcal{B}_{n+1}$, labelled as in Section 4.4.1, $\phi^{-1}(P)$ can be extracted (in reverse order) by the following procedure, the “extraction procedure”.

- (1) Treat the descents after the key D from bottom to top. (Note that the one-step descent consisting of the key D is not treated.)
- (2) To treat a descent, first record the label on its topmost D (necessarily a normal downstep). Then, if the descent has special downsteps, say $D_1, D_2, \dots, D_r$ in order top to bottom, record the label on the key D of $P(D_1), P(D_2), \dots, P(D_r)$, then perform the entire extraction procedure on $P(D_r), P(D_{r-1}), \dots, P(D_1)$ in that order.

After treating all descents, every peak downstep except the key downstep will have its label recorded. Reverse this list of labels and prepend m 0s where $m = \#UU$ s in P to get $\phi^{-1}(P)$. We give an elaborate example in Figure 10 below.

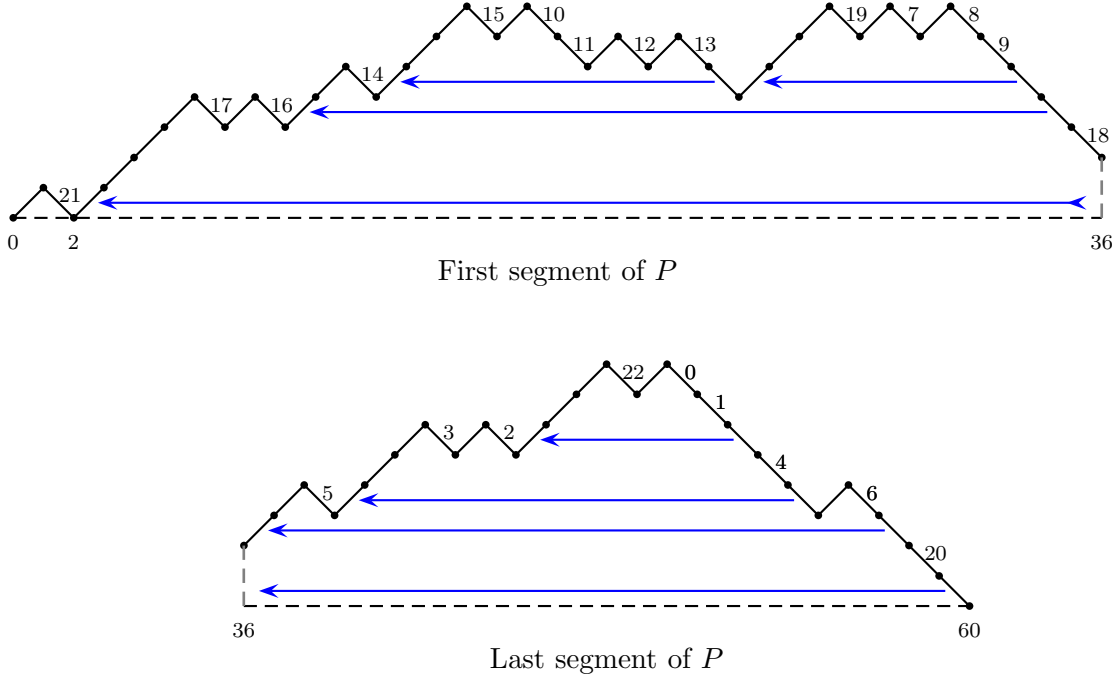


FIGURE 10. A path $P \in \mathcal{B}_{30}$, $\phi^{-1}(P) = 0^{13} \ 2 \ 5 \ 3 \ 0 \ 7 \ 10 \ 12 \ 14 \ 13 \ 16 \ 17 \ 15 \ 8 \ 21 \ 19 \ 6$

4.5. Class 95. A *left to right maximum*, *LRmax* for short, in a weak ascent sequence $a = a_1 a_2 \cdots a_n$ is an entry a_i such that $a_i > a_j$ for all $j < i$. If $a_i = m$ we say m is the value of the LRmax. Thus for $a = 011022131$, the LRmax are $a_1 = 0, a_2 = 1, a_5 = 2, a_8 = 3$. So any weak ascent sequence $a \in \mathcal{A}_n$ can be decomposed uniquely as $m_1 \pi^{(1)} \cdots m_k \pi^{(k)}$, where $m_1, \dots, m_k$ are all of the LRmax entries in a ; thus $0 = m_1 < m_2 < \cdots < m_k \leq n-1$ and $m_i > \pi^{(i)}$ (entrywise). We call this the LRmax decomposition of a .

Let $B = \{010, 101, 120\}$ and $C = \{010, 110, 120\}$. Denote the set of all words over the alphabet $\{0, 1, 2, \dots\}$ by W^* . Let $s = s_1 s_2 \cdots s_n$ be any word in W^* of length n . Let $s = m_1 \theta^{(1)} \cdots m_k \theta^{(k)}$ be the LRmax decomposition of s . Then s avoids B if and only if for all $j = 1, 2, \dots, k$,

- $m_j \geq \theta^{(j)} > m_{j-1}$, where $m_0 = 0$.
- there exists $d_j \geq 0$ such that $\theta^{(j)} = m_j^{d_j} \theta^{[j]}$ and $\theta^{[j]}$ avoids B .

Moreover, s avoids C if and only if for all $j = 1, 2, \dots, k$,

- $m_j \geq \theta^{(j)} > m_{j-1}$, where $m_0 = 0$.
- there exists $d_j \geq 0$ such that $\theta^{(j)} = \theta^{[j]} m_j^{d_j}$ and $\theta^{[j]}$ avoids C .

We define recursively $\alpha(s)$ for any word $s \in W^*$ avoids B by $m_1 \alpha(\theta^{[1]}) m_1^{d_1} \cdots m_k \alpha(\theta^{[k]}) m_k^{d_k}$. Clearly, s avoids B if and only if $\alpha(s) \in W^*$ avoids C . By restricting to weak ascent sequences, we obtain that the bijection α leads to $B \stackrel{a}{\sim} C$.

4.6. Class 105. Let $B = \{010, 100, 201\}$. Any weak ascent sequence $b = m_1\theta^{(1)} \dots m_k\theta^{(k)} \in \mathcal{A}_n(B)$ with LRmax $m_1, \dots, m_k$ satisfies

- for all $j \geq 1$, all the letters in b that are smaller than m_j are different and decreasing.
- for all $j \geq 1$, each letter of b that smaller than m_j is not belong to $\{m_1, \dots, m_{j-1}\}$.

Let $C = \{010, 100, 201\}$. Any weak ascent sequence $c = m_1\pi^{(1)} \dots m_k\pi^{(k)} \in \mathcal{A}_n(C)$ with LRmax $m_1, \dots, m_k$ satisfies

- for all $j \geq 1$, all the letters in c that are smaller than m_j are forms a nonincreasing sequence.
- for all $j \geq 1$, each letter of c that smaller than m_j is not belong to $\{m_1, \dots, m_{j-1}\}$
- for all $j \geq 1$, $\pi^{(j)}$ can be written as $m_j^{d_j}\pi^{[j]}$ with $d_j \geq 0$ and $\pi^{[j]} < m_j$.

Let $c = m_1\pi^{(1)} \dots m_k\pi^{(k)} \in \mathcal{A}_n(C)$ with LRmax $m_1, \dots, m_k$. We define $b = \alpha(c)$ to be the same c , where we rewrite any maximal sequence of type $ss \dots s = s^\ell$ with $\ell \geq 2$ in $\pi^{[j]}$ as $sm_p^{\ell-1}$ such that $m_{p-1} < s < m_p$ (here $m_0 = 0$). Clearly, $b \in \mathcal{A}_n(B)$.

Let $b = m_1\theta^{(1)} \dots m_k\theta^{(k)} \in \mathcal{A}_n(B)$ with LRmax $m_1, \dots, m_k$. We define $c = \beta(b)$ to be the same b , where we rewrite any maximal sequence of type $sm_p^{\ell-1}$ with $\ell \geq 2$ in $\theta^{(j)}$ as s^ℓ such that $m_{p-1} < s < m_p$ (here $m_0 = 0$). Clearly, $c \in \mathcal{A}_n(C)$.

Hence, there is a bijection between $\mathcal{A}_n(B)$ and $\mathcal{A}_n(C)$, which implies that $B \stackrel{a}{\sim} C$.

4.7. Class 114. Case $\{021, 100, 120\}$ By our algorithm, we obtain that the tree $\mathcal{T}(021, 100, 120)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_{m,j}, b_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1}, \\ c_{m,1} &\rightsquigarrow c_{m+1,1}, d_{m,2}, \dots, d_{m,m+1}, \\ c_{m,j} &\rightsquigarrow d_{m,j}, \dots, d_{m,m+1}, \\ d_{m,j} &\rightsquigarrow d_{m+1,j}, \dots, d_{m+1,m+2}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j 0$, and $d_{m,j} = a_m 1 j$.

Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(021, 100, 120)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, $d_{m,j}$). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{j=1}^m B_{m,j}(x), \\ B_{m,j}(x) &= x + xC_{m,j}(x) + xB_{m+1,j}(x) + x \sum_{i=j+1}^{m+1} D_{m,i}(x), \\ C_{m,1}(x) &= x + xC_{m+1,1} + x \sum_{i=2}^{m+1} D_{m,i}(x), \\ C_{m,j}(x) &= x + x \sum_{i=j}^{m+1} D_{m,i}(x), \end{aligned}$$

$$D_{m,j}(x) = x + x \sum_{i=j}^{m+2} D_{m+1,i}(x).$$

Define $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. By induction on m, j , we have that

$$D_{m,j}(x) = C_{m+1,j}(x) = xC^{m+3-j}(x).$$

Thus,

$$B_{m,j}(x) = x + xB_{m+1,j}(x) + x^2C^{m+2-j}(x) + x^2 \sum_{i=j+1}^{m+1} C^{m+3-i}(x),$$

which equivalent to

$$B_{m,j}(x) = xB_{m+1,j}(x) + xC^{m+1-j}(x)(1 + xC(x)).$$

By iterating infinity number of times (we assume $|x| < 1/4$), we obtain that $B_{m,j}(x) = x(1 + xC(x))C^{m+2-j}(x)$. Hence, by the recurrence of $A_m(x)$, we have

$$A_1(x) = \frac{x}{1-x} + x \sum_{m \geq 1} \sum_{j=1}^m (1 + xC(x))x^m C^{m+2-j}(x),$$

which implies

$$A_1(x) = x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}.$$

Hence, the generating function for the number of weak ascent sequences of length $n \geq 0$ that avoid $\{021, 100, 120\}$ is given by $1 + A_1(x) = 1 + x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}$, as required.

Case $\{021, 101, 120\}$ By our algorithm, we obtain that the tree $\mathcal{T}(021, 100, 120)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_{m,j}, b_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1}, \\ c_{m,j} &\rightsquigarrow c_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1}, \\ d_{m,j} &\rightsquigarrow d_{m+1,j}, \dots, d_{m+1,m+2}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j 0$, and $d_{m,j} = a_m 1 j$. Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(021, 100, 120)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, $d_{m,j}$). Thus,

$$A_m(x) = x + xA_{m+1}(x) + x \sum_{j=1}^m B_{m,j}(x),$$

$$B_{m,j}(x) = x + xC_{m,j}(x) + xB_{m+1,j}(x) + x \sum_{i=j+1}^{m+1} D_{m,i}(x),$$

$$C_{m,j}(x) = x + xC_{m+1,j}(x) + x \sum_{i=j+1}^{m+1} D_{m,i}(x),$$

$$D_{m,j}(x) = x + x \sum_{i=j}^{m+2} D_{m+1,i}(x).$$

As in the previous case, by induction on m, j , we have that

$$D_{m,j}(x) = C_{m+1,j}(x) = xC^{m+3-j}(x).$$

Then, we obtain that $B_{m,j}(x) = x(1 + xC(x))C^{m+2-j}(x)$. Hence, by the recurrence of $A_m(x)$, we have

$$A_1(x) = x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}.$$

Hence, the generating function for the number of weak ascent sequences of length $n \geq 0$ that avoid $\{021, 101, 120\}$ is given by $1 + A_1(x) = 1 + x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}$, as required.

Case $\{021, 110, 120\}$ By our algorithm, we obtain that the tree $\mathcal{T}(021, 100, 120)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_{m,j}, d_{m,j}, \dots, d_{m,m+1}, \\ c_{m,j} &\rightsquigarrow c_{m+1,j}, d_{m,j}, \dots, d_{m,m+1}, \\ d_{m,j} &\rightsquigarrow d_{m+1,j}, \dots, d_{m+1,m+2}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j 0$, and $d_{m,j} = a_m 1 j$. Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(021, 100, 120)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, $d_{m,j}$). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{j=1}^m B_{m,j}(x), \\ B_{m,j}(x) &= x + xC_{m,j}(x) + x \sum_{i=j}^{m+1} D_{m,i}(x), \\ C_{m,j}(x) &= x + xC_{m+1,j}(x) + x \sum_{i=j}^{m+1} D_{m,i}(x), \\ D_{m,j}(x) &= x + x \sum_{i=j}^{m+2} D_{m+1,i}(x). \end{aligned}$$

As in the previous case, by induction on m, j , we have that

$$D_{m,j}(x) = C_{m,j}(x) = xC^{m+3-j}(x).$$

Then, we obtain that $B_{m,j}(x) = x(1 + xC(x))C^{m+2-j}(x)$. Hence, by the recurrence of $A_m(x)$, we have

$$A_1(x) = x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}.$$

Hence, the generating function for the number of weak ascent sequences of length $n \geq 0$ that avoid $\{021, 101, 120\}$ is given by $1 + A_1(x) = 1 + x(1 + xC(x))C^2(x) - \frac{x^2C(x)}{1-x}$, as required.

4.8. **Class 140.** Let $B = \{110, 120, 201\}$. Any weak ascent sequence $b = m_1\theta^{(1)} \cdots m_k\theta^{(k)} \in \mathcal{A}_n(B)$ with LRmax $m_1, \dots, m_k$ satisfies

- for all $j \geq 1$, $m_j \geq \theta^{(j)} \geq m_{j-1}$, where $m_0 = 0$.
- removing all occurrences of m_j from $\theta^{(j)}$, we obtain a nonincreasing sequence.
- $\theta^{(j)}$ can be written as $\theta^{[j]}m_j \cdots m_j$ such that $\theta^{[j]} < m_j$.

Let $C = \{100, 120, 201\}$. Any weak ascent sequence $a = m_1\pi^{(1)} \cdots m_k\pi^{(k)} \in \mathcal{A}_n(C)$ with LRmax $m_1, \dots, m_k$ satisfies

- for all $j \geq 1$, $m_j \geq \pi^{(j)} \geq m_{j-1}$, where $m_0 = 0$.
- removing all occurrences of m_j from $\pi^{(j)}$, we obtain a decreasing sequence.

To show that $B \stackrel{a}{\sim} C$, we present a bijection α between the set of weak ascent sequences in $\mathcal{A}_n(B)$ with LRmax $m_1, \dots, m_k$ and the set of weak ascent sequences in $\mathcal{A}_n(C)$ with LRmax $m_1, \dots, m_k$. By the above two LRmax decompositions, to define α , we have to map bijectively $\theta^{(j)}$ to $\pi^{(j)}$. Let $\theta^{(j)} = r_1^{s_1}r_2^{s_2} \cdots r_t^{s_t}m_j^k$, where $s_1, \dots, s_t \geq 1$, $t \geq 0$, $m_j > r_1 > \cdots > r_t \geq m_{j-1}$, and $k \geq 0$. We define $\alpha(\theta^{(j)}) = m_j^{s_1-1}s_1m_j^{s_2-1}s_2 \cdots m_j^{s_t-1}s_tm_j^k$. Note that $\theta^{(j)}$ avoids B and $\pi^{(j)}$ avoids C . Hence, by extending α to all $\theta^{(j)}$, we obtain that α is a bijection as required, which implies $B \stackrel{a}{\sim} C$.

5. OPEN CASES SUMMARIZED

Recall WAS_k is the number of A -Wilf equivalence classes for weak ascent sequences avoiding k distinct patterns of length three.

In Section 3, we showed that $59 \leq \text{WAS}_2 \leq 61$. To establish Conjecture 1.2 for pairs, one must show that (see Classes 41 and 43 in Table 1)

$$\{102, 110\} \stackrel{a}{\sim} \{102, 120\} \text{ and } \{101, 102\} \stackrel{a}{\sim} \{021\}.$$

Note that $|\mathcal{A}_n(102, 110)|$ is given by sequence A279561.

In Section 4, we showed that $150 \leq \text{WAS}_3 \leq 155$. To establish Conjecture 1.2 for triples, one must show that (see Classes 120, 123, 132, and 139 in Table 3)

$$\begin{aligned} \{102, 110, 120\} &\stackrel{a}{\sim} \{021, 120\}, \\ \{101, 102, 110\} &\stackrel{a}{\sim} \{021, 100\}, \text{ so both counted by } \binom{2n}{n}, \\ \{102, 110, 210\} &\stackrel{a}{\sim} \{102, 120, 210\}, \\ \{100, 110, 201\} &\stackrel{a}{\sim} \{101, 110, 201\} \stackrel{a}{\sim} \{021\}. \end{aligned}$$

6. FURTHER RESULTS

At first glance, one might wonder whether the algorithm for generating trees can aid in finding the generating function for weak ascent sequences in $\mathcal{A}_n(P)$, where P is a trivial class. Below, we provide (only) two examples of such trivial classes of pattern pairs. Naturally, questions about the other trivial classes may arise, but these will be addressed in a future work.

Theorem 6.1. (1) *The generating function for the number of weak ascent sequences in $\mathcal{A}_n(021, 100)$ is given by*

$$\frac{x}{\sqrt{1-4x}}.$$

(2) *The generating function for the number of weak ascent sequences in $\mathcal{A}_n(021, 120)$ is given by*

$$\frac{-2 + 9x - 7x^2 - 2x^3 + (2 - 3x - x^2)\sqrt{1-4x}}{2(1-x)(1-4x-x^2)}.$$

Proof. (1) By our algorithm, we obtain that the tree $\mathcal{T}(021, 100)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_{m,j}, b_{m+1,j}, \dots, b_{m+1,m+1}, \\ c_{m,j} &\rightsquigarrow c_{m+1,j}, \dots, c_{m+1,m+1}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, and $c_{m,j} = a_m j 0$. Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(021, 100)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$). Thus,

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^m B_{m,i}(x), \\ B_{m,j}(x) &= x + xC_{m,j}(x) + x \sum_{i=j}^{m+1} B_{m+1,i}(x), \\ C_{m,j}(x) &= x + x \sum_{i=j}^{m+1} C_{m+1,i}(x). \end{aligned}$$

Define $A(x; v) = \sum_{m \geq 1} A_m(x) v^{m-1}$, $B(x; v, u) = \sum_{m \geq 1} \sum_{j=1}^m B_{m,j}(x) v^{m-1} u^{m-j}$, and $C(x; v, u) = \sum_{m \geq 1} \sum_{j=1}^m C_{m,j}(x) v^{m-1} u^{m-j}$. Thus, the above recurrences can be written as

$$\begin{aligned} A(x; v) &= \frac{x}{1-v} + \frac{x}{v}(A(x; v) - A(x; 0)) + xB(x; v, 1), \\ B(x; v, u) &= \frac{x}{(1-v)(1-vu)} + xC(x; v, u) + \frac{x}{vu(1-u)}(B(x; v, u) - uB(x; vu, 1) - (1-u)B(x; v, 0)), \\ C(x; v, u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-u)}(C(x; v, u) - uC(x; vu, 1) - (1-u)C(x; v, 0)). \end{aligned}$$

It is not hard to check that

$$\begin{aligned} C(x; v, u) &= \frac{(uv-1)\sqrt{1-4x} - uv - 2x + 1}{2(1-v)(u^2v^2 - uv + x)}, \\ B(x; v, u) &= -\frac{(u^3v^3 + 2(x^2 + x - 1)u^2v^2 - (3x^2 + x - 1)uv + 3x^2 - x)}{2(u^2v^2 - uv + x)^2(1-v)} \\ &\quad - \frac{(uv-1)((2x^2 + 4x - 1)uv(uv-1) - 2x^3 + 5x^2 - x)}{2(u^2v^2 - uv + x)^2\sqrt{1-4x}(1-v)}. \end{aligned}$$

Hence, by taking $v = x$, we obtain that

$$A(x; 0) = \frac{x}{1-x} + \frac{x(1-x-\sqrt{1-4x})}{(1-x)\sqrt{1-4x}} = \frac{x}{\sqrt{1-4x}},$$

as claimed.

(1) By our algorithm, we obtain that the tree $\mathcal{T}(021, 120)$ with a root a_1 has the following rules

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow c_{m,j}, b_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1}, \\ c_{m,j} &\rightsquigarrow c_{m+1,j}, b_{m+1,j}, d_{m,j+1}, \dots, d_{m,m+1}, \\ d_{m,j} &\rightsquigarrow d_{m+1,j}, \dots, d_{m+1,m+2}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j 0$, and $d_{m,j} = a_m 1 j$. Let $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_{m,j}(x)$) be the generating function for the number of nodes at level $n \geq 1$ in the subtree of the tree $\mathcal{T}(021, 100)$ with a root a_m (respectively, $b_{m,j}$, $c_{m,j}$, $d_{m,j}$). Define $A(x; v) = \sum_{m \geq 1} A_m(x) v^{m-1}$, $B(x; v, u) = \sum_{m \geq 1} \sum_{j=1}^m B_{m,j}(x) v^{m-1} u^{m-j}$, $C(x; v, u) = \sum_{m \geq 1} \sum_{j=1}^m C_{m,j}(x) v^{m-1} u^{m-j}$, and $D(x; v, u) = \sum_{m \geq 1} \sum_{j=2}^{m+1} D_{m,j}(x) v^{m-1} u^{m+1-j}$. Thus, the rules of $\mathcal{T}(021, 120)$ can be written as

$$\begin{aligned} A(x; v) &= \frac{x}{1-v} + \frac{x}{v}(A(x; v) - A(x; 0)) + xB(x; v, 1), \\ B(x; v, u) &= \frac{x}{(1-v)(1-vu)} + xC(x; v, u) + \frac{x}{uv}(B(x; v, u) - B(x; v, 0)) \\ &\quad + \frac{x}{1-u}(D(x; v, u) - uD(x; vu, 1)), \\ C(x; v, u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{uv}(B(x; v, u) + C(x; v, u) - B(x; v, 0) - C(x; v, 0)) \\ &\quad + \frac{x}{1-u}(D(x; v, u) - uD(x; vu, 1)), \\ D(x; v, u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-u)}(D(x; v, u) - uD(x; vu, 1)) - \frac{x}{vu}D(x; v, 0). \end{aligned}$$

It is not hard to check that

$$\begin{aligned} D(x; v, u) &= \frac{1 - 2x - uv + (uv - 1)\sqrt{1-4x}}{2(1-v)(u^2v^2 - uv + x)}, \\ C(x; v, u) &= \frac{(1+x)\sqrt{1-4x} + 3x - 1}{(1-v)(1-4x-x^2)(1-2vu+\sqrt{1-4x})}, \\ B(x; v, u) &= \frac{((x+3)\sqrt{1-4x} + x - 1)x}{(1-v)(1-4x-x^2)(1-2vu+\sqrt{1-4x})}. \end{aligned}$$

Hence, by taking $v = x$, we obtain that

$$A(x; 0) = \frac{x}{1-x} + xB(x; x, 1) = \frac{-2 + 9x - 7x^2 - 2x^3 + (2 - 3x - x^2)\sqrt{1-4x}}{2(1-x)(1-4x-x^2)},$$

as claimed. $\square$

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APPENDIX

Table 3: Weak ascent sequences avoiding a pair of length-3 patterns.

Beginning of Table 3		
Class	B triple	$\{ \mathcal{A}_n(B) \}_{n=1}^1$
1	$\{001,010\}, \{001,011\}, \{001,012\}$	1,2,3,4,5,6,7,8,9,10,11
2	$\{000,001\}$	1,2,3,5,8,13,21,34,55,89,144
3	$\{000,010\}$	1,2,4,10,28,85,277,964,3549,13737,55684
4	$\{000,012\}$	1,2,4,5,2,1,0,0,0,0,0
5	$\{000,011\}, \{001,021\}, \{001,110\}$ $\{001,120\}$	1,2,4,7,11,16,22,29,37,46,56
6	$\{001,100\}$	1,2,4,7,12,20,33,54,88,143,232
7	$\{001,210\}$	1,2,4,8,15,26,42,64,93,130,176
8	$\{001,101\}, \{001,102\}, \{001,201\}$ $\{010,012\}, \{011,012\}$	1,2,4,8,16,32,64,128,256,512,1024
9	$\{010,011\}$	1,2,4,9,22,58,164,494,1577,5311,18794
10	$\{012,100\}$	1,2,5,12,27,56,110,207,378,675,1186
11	$\{012,021\}, \{012,110\}$	1,2,5,12,27,58,121,248,503,1014,2037
12	$\{012,101\}$	1,2,5,12,28,65,151,351,816,1897,4410
13	$\{012,201\}, \{012,210\}$	1,2,5,13,33,80,185,411,885,1862,3853
14	$\{000,021\}$	1,2,5,13,33,87,232,629,1724,4772,13313
15	$\{011,102\}, \{012,102\}, \{012,120\}$	1,2,5,13,34,89,233,610,1597,4181,10946
16	$\{011,021\}$	1,2,5,13,35,97,275,794,2327,6905,20705
17	$\{011,120\}$	1,2,5,13,35,97,275,795,2335,6953,20948
18	$\{011,100\}$	1,2,5,13,36,106,330,1083,3734,13483,50847
19	$\{000,102\}$	1,2,5,14,40,121,373,1181,3796,12391,40902
20	$\{000,120\}$	1,2,5,14,41,128,415,1397,4839,17207,62561
21	$\{010,021\}, \{011,201\}$	1,2,5,14,42,132,429,1430,4862,16796,58786
22	$\{011,210\}$	1,2,5,14,42,133,440,1508,5320,19225,70896
23	$\{000,110\}$	1,2,5,14,43,141,489,1780,6763,26715,109372
24	$\{011,101\}, \{011,110\}$	1,2,5,14,43,143,510,1936,7775,32869,145665
25	$\{000,101\}$	1,2,5,14,44,150,548,2127,8703,37340,167266
26	$\{000,201\}$	1,2,5,15,50,182,704,2866,12147,53271,240400
27	$\{000,210\}$	1,2,5,15,50,183,717,2982,13028,59417,281255
28	$\{000,100\}$	1,2,5,15,50,185,742,3205,14768,72176,372183
29	$\{010,102\}$	1,2,5,15,51,186,707,2763,11024,44714,183830
30	$\{010,120\}$	1,2,5,15,51,189,746,3093,13350,59607,274013
31	$\{010,110\}$	1,2,5,15,51,189,747,3107,13484,60705,282302
32	$\{010,100\}$	1,2,5,15,51,191,775,3367,15530,75559,385802
33	$\{010,101\}$	1,2,5,15,51,191,775,3368,15549,75788,388083
34	$\{010,201\}$	1,2,5,15,52,201,842,3750,17538,85402,430284
35	$\{010,210\}$	1,2,5,15,52,201,843,3768,17741,87271,445739
36	$\{021,102\}$	1,2,6,20,66,213,683,2211,7291,24552,84305
37	$\{021,110\}$	1,2,6,20,69,242,857,3056,10959,39493,142955
38	$\{021,120\}$	1,2,6,20,69,242,859,3080,11140,40596,148904
39	$\{021,100\}$	1,2,6,20,70,252,924,3432,12870,48620,184756
40	$\{021,101\}$	1,2,6,20,71,264,1015,4002,16094,65758,272208
41	$\{102,110\}, \{102,120\}$	1,2,6,21,77,287,1079,4082,15522,59280,227240
42	$\{100,102\}$	1,2,6,21,80,318,1305,5487,23535,102603,453400
43	$\{021,201\}, \{021,210\}, \{101,102\}$	1,2,6,21,80,322,1347,5798,25512,114236,518848
44	$\{101,120\}$	1,2,6,21,80,323,1365,5990,27128,126183,600506
45	$\{100,120\}$	1,2,6,21,81,333,1437,6437,29705,140428,677247
46	$\{110,120\}$	1,2,6,21,81,335,1461,6646,31295,151700,753824
47	$\{101,110\}$	1,2,6,21,81,336,1478,6832,32976,165428,859527
48	$\{100,110\}$	1,2,6,21,82,345,1537,7169,34739,173924,895993
49	$\{100,101\}$	1,2,6,21,83,359,1668,8227,42697,231671,1307753
50	$\{102,210\}$	1,2,6,22,87,351,1416,5681,22660,89961,355924
51	$\{102,201\}$	1,2,6,22,87,354,1465,6154,26223,113236,494870
52	$\{110,201\}$	1,2,6,22,90,394,1807,8575,41768,207678,1049909
53	$\{120,201\}$	1,2,6,22,90,394,1811,8645,42517,214090,1098516
54	$\{120,210\}$	1,2,6,22,90,396,1841,8940,44965,232747,1233821
55	$\{101,210\}$	1,2,6,22,91,409,1956,9824,51359,277692,1545338
56	$\{110,210\}$	1,2,6,22,91,410,1973,10012,53094,292273,1661525
57	$\{100,201\}, \{101,201\}$	1,2,6,22,92,422,2074,10754,58202,326240,1882960
58	$\{100,210\}$	1,2,6,22,92,424,2106,11102,61436,353980,2110366
59	$\{201,210\}$	1,2,6,23,102,495,2549,13682,75714,428882,2474573
End of Table 3		

Table 4: Weak ascent sequences avoiding a triple of length-3 patterns.

Beginning of Table 4		
Class	B triple	$\{ \mathcal{A}_n(B) \}_{n=1}^{10}$
1	{000,001,012}	1,2,2,1,0,0,0,0,0,0
2	{000,001,010}, {000,001,011} {001,010,011}, {001,010,012} {001,011,012}	1,2,2,2,2,2,2,2,2,2
3	{000,011,012}	1,2,3,1,0,0,0,0,0,0
4	{000,010,012}	1,2,3,3,2,1,0,0,0,0
5	{001,011,100}, {001,011,120} {001,012,100}, {001,012,110}	1,2,3,3,3,3,3,3,3,3
6	{000,001,021}, {000,001,120}	1,2,3,4,4,4,4,4,4,4
7	{000,001,110}, {001,010,021} {001,010,100}, {001,010,101} {001,010,102}, {001,010,110} {001,010,120}, {001,010,201} {001,010,210}, {001,011,021} {001,011,101}, {001,011,102} {001,011,110}, {001,011,201} {001,011,210}, {001,012,021} {001,012,101}, {001,012,102} {001,012,120}, {001,012,201} {001,012,210}	1,2,3,4,5,6,7,8,9,10
8	{000,001,210}	1,2,3,5,7,9,11,13,15,17
9	{000,010,011}	1,2,3,5,8,12,17,23,30,38
10	{000,001,100}, {000,001,101} {000,001,102}, {000,001,201} {010,011,012}	1,2,3,5,8,13,21,34,55,89
11	{000,010,102}	1,2,4,10,27,73,204,587,1716,5085
12	{000,010,120}	1,2,4,10,27,76,224,686,2161,6977
13	{000,010,110}	1,2,4,10,27,77,229,708,2260,7414
14	{000,010,100}	1,2,4,10,27,78,241,786,2686,9587
15	{000,010,101}	1,2,4,10,27,79,247,819,2859,10461
16	{000,010,201}	1,2,4,10,28,85,275,936,3321,12203
17	{000,010,210}	1,2,4,10,28,85,275,938,3347,12421
18	{000,012,021}	1,2,4,4,0,0,0,0,0,0
19	{000,012,110}	1,2,4,4,1,0,0,0,0,0
20	{000,012,101}	1,2,4,4,2,1,0,0,0,0
21	{000,012,100}, {000,012,102} {000,012,120}, {000,012,201} {000,012,210}	1,2,4,5,2,1,0,0,0,0
22	{000,011,021}, {000,011,102} {000,011,120}, {001,021,100} {001,021,110}, {001,021,120} {001,100,110}, {001,100,120} {001,110,120}	1,2,4,6,8,10,12,14,16,18
23	{000,011,100}, {000,011,101} {000,011,110}, {000,011,201} {000,011,210}, {001,021,101} {001,021,102}, {001,021,201} {001,021,210}, {001,100,210} {001,101,110}, {001,101,120} {001,102,110}, {001,102,120} {001,110,201}, {001,110,210} {001,120,201}, {001,120,210} {010,012,021}, {011,012,021}	1,2,4,7,11,16,22,29,37,46
24	{001,100,101}, {001,100,102} {001,100,201}	1,2,4,7,12,20,33,54,88,143
25	{011,012,100}	1,2,4,7,13,23,41,72,126,219
26	{001,101,210}, {001,102,210} {001,201,210}, {011,012,210}	1,2,4,8,15,26,42,64,93,130
27	{010,012,100}, {010,012,110} {011,012,201}	1,2,4,8,15,27,47,80,134,222
28	{010,012,210}	1,2,4,8,16,31,57,99,163,256
29	{001,101,102}, {001,101,201} {001,102,201}, {010,011,021} {010,012,101}, {010,012,102} {010,012,120}, {010,012,201} {011,012,101}, {011,012,102} {011,012,110}, {011,012,120}	1,2,4,8,16,32,64,128,256,512
30	{010,011,102}	1,2,4,9,21,51,126,316,799,2034
31	{000,010,021}	1,2,4,9,21,51,127,323,835,2188
32	{010,011,120}	1,2,4,9,21,51,127,324,842,2225

Continuation of Table 4		
Class	B triple	$\{ \mathcal{A}_n(B) \}_{n=1}^{11}$
33	{010,011,201}	1,2,4,9,22,57,154,429,1223,3550
34	{010,011,210}	1,2,4,9,22,57,155,439,1286,3875
35	{010,011,100}, {010,011,101} {010,011,110}	1,2,4,9,22,58,164,494,1577,5311
36	{012,021,100}, {012,021,101} {012,021,110}	1,2,5,11,21,36,57,85,121,166
37	{012,100,110}	1,2,5,11,22,39,66,108,175,283
38	{012,100,101}	1,2,5,11,23,45,85,153,281,499
39	{012,101,110}	1,2,5,11,23,47,95,191,383,767
40	{000,021,102}	1,2,5,12,25,60,148,374,962,2511
41	{012,100,210}, {012,110,210}	1,2,5,12,26,51,92,155,247,376
42	{012,100,201}, {012,110,201}	1,2,5,12,26,51,93,161,269,439
43	{012,100,102}, {012,100,120}	1,2,5,12,27,56,110,207,378,675
44	{012,101,210}	1,2,5,12,27,57,113,211,373,628
45	{011,021,102}, {012,021,102} {012,021,120}, {012,021,201} {012,021,210}, {012,101,201} {012,102,110}, {012,110,120}	1,2,5,12,27,58,121,248,503,1014
46	{000,021,120}	1,2,5,12,27,65,160,404,1038,2707
47	{011,021,100}, {011,021,120} {011,102,120}	1,2,5,12,28,64,144,320,704,1536
48	{012,101,102}, {012,101,120}	1,2,5,12,28,65,151,351,816,1897
49	{000,021,110}	1,2,5,12,28,67,164,411,1050,2726
50	{011,100,102}	1,2,5,12,30,75,190,483,1235,3167
51	{011,100,120}	1,2,5,12,30,76,197,518,1383,3737
52	{000,021,101}	1,2,5,12,30,77,202,539,1458,3988
53	{012,201,210}	1,2,5,13,32,73,156,318,629,1224
54	{000,102,120}	1,2,5,13,32,85,223,599,1617,4410
55	{012,102,201}, {012,102,210} {012,120,201}, {012,120,210}	1,2,5,13,33,80,185,411,885,1862
56	{011,102,210}	1,2,5,13,33,81,193,449,1025,2305
57	{011,102,201}	1,2,5,13,33,82,201,489,1185,2866
58	{000,021,100}, {000,021,201} {000,021,210}	1,2,5,13,33,87,232,629,1724,4772
59	{011,120,201}	1,2,5,13,34,89,233,610,1595,4161
60	{011,101,102}, {011,102,110} {012,102,120}	1,2,5,13,34,89,233,610,1597,4181
61	{011,120,210}	1,2,5,13,34,90,241,651,1769,4826
62	{000,102,110}	1,2,5,13,34,91,246,672,1850,5127
63	{011,021,101}, {011,021,110} {011,021,201}, {011,021,210} {011,100,201}	1,2,5,13,35,97,275,794,2327,6905
64	{011,101,120}, {011,110,120}	1,2,5,13,35,97,275,795,2335,6953
65	{011,100,210}	1,2,5,13,35,98,284,847,2589,8085
66	{000,101,120}	1,2,5,13,36,105,318,993,3183,10435
67	{000,101,110}	1,2,5,13,36,105,318,995,3205,10591
68	{011,100,101}, {011,100,110}	1,2,5,13,36,106,330,1083,3734,13483
69	{000,101,102}	1,2,5,13,37,108,327,1010,3180,10155
70	{000,110,120}	1,2,5,13,37,109,336,1067,3484,11640
71	{000,102,210}	1,2,5,14,39,113,325,945,2747,8012
72	{000,100,102}	1,2,5,14,39,115,347,1069,3351,10648
73	{000,102,201}	1,2,5,14,39,116,345,1060,3289,10371
74	{000,100,120}	1,2,5,14,40,121,377,1206,3943,13120
75	{011,201,210}	1,2,5,14,41,123,375,1157,3603,11304
76	{000,120,201}	1,2,5,14,41,127,405,1331,4465,15250
77	{000,120,210}	1,2,5,14,41,127,407,1349,4587,15956
78	{010,021,100}, {010,021,101} {010,021,102}, {010,021,110} {010,021,120}, {010,021,201} {010,021,210}, {011,101,201} {011,110,201}	1,2,5,14,42,132,429,1430,4862,16796
79	{000,110,201}	1,2,5,14,42,132,429,1431,4872,16863
80	{011,101,210}, {011,110,210}	1,2,5,14,42,133,440,1508,5320,19225
81	{000,100,110}	1,2,5,14,42,134,448,1560,5627,20942
82	{000,110,210}	1,2,5,14,43,140,480,1717,6374,24450
83	{000,101,210}	1,2,5,14,43,141,488,1767,6650,25882
84	{000,100,101}	1,2,5,14,43,143,504,1870,7253,29241
85	{011,101,110}	1,2,5,14,43,143,510,1936,7775,32869
86	{000,101,201}	1,2,5,14,44,148,528,1972,7647,30605
87	{000,201,210}	1,2,5,15,49,171,621,2328,8935,34956

Continuation of Table 4		
Class	B triple	$\{(\mathcal{A}_n(B))\}_{n=1}^{11}$
88	{000,100,201}	1,2,5,15,49,173,644,2506,10101,41933
89	{000,100,210}	1,2,5,15,49,174,657,2616,10884,47028
90	{010,102,120}	1,2,5,15,50,175,627,2277,8347,30822
91	{010,102,110}	1,2,5,15,50,175,628,2289,8436,31358
92	{010,100,102}	1,2,5,15,50,177,650,2449,9410,36733
93	{010,101,102}	1,2,5,15,50,178,662,2540,9977,39916
94	{010,100,120}	1,2,5,15,50,179,674,2637,10635,43965
95	{010,101,120}, {010,110,120}	1,2,5,15,50,179,675,2650,10749,44805
96	{010,100,110}	1,2,5,15,50,179,676,2662,10849,45528
97	{010,101,110}	1,2,5,15,50,179,677,2678,11009,46827
98	{010,100,101}	1,2,5,15,50,181,702,2889,12533,57027
99	{010,102,210}	1,2,5,15,51,185,692,2629,10076,38830
100	{010,102,201}	1,2,5,15,51,185,693,2648,10277,40427
101	{010,110,201}	1,2,5,15,51,188,731,2951,12254,52030
102	{010,120,201}	1,2,5,15,51,188,731,2955,12317,52634
103	{010,110,210}	1,2,5,15,51,188,733,2983,12570,54544
104	{010,120,210}	1,2,5,15,51,188,733,2984,12583,54641
105	{010,100,201}, {010,101,201}	1,2,5,15,51,190,759,3203,14129,64645
106	{010,100,210}	1,2,5,15,51,190,760,3219,14292,66014
107	{010,101,210}	1,2,5,15,51,190,760,3220,14311,66236
108	{010,201,210}	1,2,5,15,52,200,825,3570,15995,73606
109	{021,102,110}	1,2,6,19,57,168,506,1585,5165,17382
110	{021,102,120}	1,2,6,19,58,174,528,1649,5328,17764
111	{021,100,102}	1,2,6,19,59,183,580,1893,6347,21741
112	{021,101,102}	1,2,6,19,60,191,619,2048,6909,23704
113	{021,100,110}	1,2,6,19,61,198,651,2171,7345,25194
114	{021,100,120}, {021,101,120} {021,110,120}	1,2,6,19,61,199,661,2234,7668,26674
115	{021,101,110}	1,2,6,19,62,207,703,2420,8424,29602
116	{021,100,101}	1,2,6,19,63,216,759,2717,9867,36244
117	{021,102,201}, {021,102,210}	1,2,6,20,66,213,683,2211,7291,24552
118	{100,102,120}	1,2,6,20,69,240,842,2979,10628,38211
119	{021,110,201}, {021,110,210}	1,2,6,20,69,242,857,3056,10959,39493
120	{021,120,201}, {021,120,210} {102,110,120}	1,2,6,20,69,242,859,3080,11140,40596
121	{101,102,120}	1,2,6,20,69,243,869,3145,11491,42312
122	{100,102,110}	1,2,6,20,70,248,891,3236,11866,43862
123	{021,100,201}, {021,100,210} {101,102,110}	1,2,6,20,70,252,924,3432,12870,48620
124	{021,101,201}, {021,101,210}	1,2,6,20,71,264,1015,4002,16094,65758
125	{100,101,120}	1,2,6,20,72,272,1067,4312,17848,75330
126	{101,110,120}	1,2,6,20,72,274,1088,4470,18886,81693
127	{100,101,110}	1,2,6,20,72,275,1098,4545,19395,84977
128	{100,101,102}	1,2,6,20,73,280,1116,4572,19140,81510
129	{100,110,120}	1,2,6,20,73,281,1128,4671,19829,85881
130	{102,110,201}	1,2,6,21,75,267,951,3404,12268,44530
131	{102,120,201}	1,2,6,21,76,274,979,3479,12351,43951
132	{102,110,210}, {102,120,210}	1,2,6,21,76,276,1002,3641,13261,48451
133	{100,102,210}	1,2,6,21,78,296,1133,4356,16797,64923
134	{101,102,210}	1,2,6,21,78,297,1144,4433,17238,67184
135	{100,102,201}	1,2,6,21,78,299,1176,4729,19378,80667
136	{101,102,201}	1,2,6,21,79,311,1265,5275,22431,96900
137	{101,120,201}	1,2,6,21,79,311,1270,5347,23074,101544
138	{101,120,210}	1,2,6,21,79,312,1283,5459,23886,106912
139	{021,201,210}, {100,110,201} {101,110,201}	1,2,6,21,80,322,1347,5798,25512,114236
140	{100,120,201}, {110,120,201}	1,2,6,21,80,322,1350,5837,25843,116581
141	{100,120,210}	1,2,6,21,80,324,1374,6036,27263,125931
142	{110,120,210}	1,2,6,21,80,324,1375,6052,27425,127273
143	{101,110,210}	1,2,6,21,80,324,1377,6087,27807,130643
144	{100,110,210}	1,2,6,21,81,334,1447,6514,30241,143993
145	{100,101,210}	1,2,6,21,81,335,1459,6615,30980,149026
146	{100,101,201}	1,2,6,21,82,346,1547,7236,35090,175268
147	{102,201,210}	1,2,6,22,85,328,1253,4754,17994,68158
148	{110,201,210}	1,2,6,22,89,382,1708,7871,37125,178363
149	{120,201,210}	1,2,6,22,89,383,1727,8072,38777,190292
150	{100,201,210}, {101,201,210}	1,2,6,22,90,394,1806,8558,41586,206098
End of Table 4		

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