Lecture 12: Relationship among convergence modes and uniform integrability

Theorem 1.8. Let $X, X_1, X_2, \ldots$ be random k-vectors.
(i) If $X_n \to_{a.s.} X$, then $X_n \to_p X$. (The converse is not true.)
(ii) If $X_n \to_{L_r} X$ for an $r > 0$, then $X_n \to_p X$. (The converse is not true.)
(iii) If $X_n \to_p X$, then $X_n \to_d X$. (The converse is not true.)
(iv) (Skorohod's theorem). If $X_n \to_d X$, then there are random vectors $Y, Y_1, Y_2, \ldots$ defined on a common probability space such that $P_Y = P_X$, $P_{Y_n} = P_{X_n}$, $n = 1, 2, \ldots$, and $Y_n \to_{a.s.} Y$. (A useful result; a conditional converse of (i)-(iii).)
(v) If, for every $\epsilon > 0$, $\sum_{n=1}^{\infty} P(\|X_n - X\| \geq \epsilon) < \infty$, then $X_n \to_{a.s.} X$. (A conditional converse of (i): $P(\|X_n - X\| \geq \epsilon)$ tends to 0 fast enough.)
(vi) If $X_n \to_p X$, then there is a subsequence $\{X_{n_j}, j = 1, 2, \ldots\}$ such that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$. (A partial converse of (i).)
(vii) If $X_n \to_d X$ and $P(X = c) = 1$, where $c \in \mathcal{R}^k$ is a constant vector, then $X_n \to_p c$. (A conditional converse of (i).)
(viii) Suppose that $X_n \to_d X$. Then, for any $r > 0$,

$$\lim_{n\to\infty} E\|X_n\|_r^r = E\|X\|_r^r < \infty \tag{1}$$

if and only if $\{\|X_n\|_r^r\}$ is *uniformly integrable* in the sense that

$$\lim_{t\to\infty} \sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}}\right) = 0. \tag{2}$$

(A conditional converse of (ii).)

Discussion on uniform integrability
If there is only one random vector, then (2) is

$$\lim_{t\to\infty} E\left(\|X\|_r^r I_{\{\|X\|_r > t\}}\right) = 0,$$

which is equivalent to the integrability of $\|X\|_r^r$ (dominated convergence theorem).
Sufficient conditions for uniform integrability:

$$\sup_n E\|X_n\|_r^{r+\delta} < \infty \quad \text{for a } \delta > 0$$

This is because

$$\begin{aligned}
\lim_{t\to\infty} \sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}}\right) &\leq \lim_{t\to\infty} \sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}} \frac{\|X_n\|_r^\delta}{t^\delta}\right) \\
&\leq \lim_{t\to\infty} \frac{1}{t^\delta} \sup_n E\left(\|X_n\|_r^{r+\delta}\right) \\
&= 0
\end{aligned}$$

Exercises 117-120.

Proof of Theorem 1.8. (i) The result follows from Lemma 1.4.
(ii) The result follows from Chebyshev's inequality with $\varphi(t) = |t|^r$.
(iii) Assume $k = 1$. (The general case is proved in the textbook.)
Let x be a continuity point of F_X and $\epsilon > 0$ be given. Then

$$\begin{aligned} F_X(x-\epsilon) &= P(X \leq x-\epsilon) \\ &\leq P(X_n \leq x) + P(X \leq x-\epsilon, X_n > x) \\ &\leq F_{X_n}(x) + P\left(|X_n - X| > \epsilon\right). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$F_X(x-\epsilon) \leq \liminf_n F_{X_n}(x).$$

Switching X_n and X in the previous argument, we can show that

$$F_X(x+\epsilon) \geq \limsup_n F_{X_n}(x).$$

Since ϵ is arbitrary and F_X is continuous at x, $F_X(x) = \lim_{n\to\infty} F_{X_n}(x)$.
(iv) The proof of this part can be found in Billingsley (1986, pp. 399-402).
(v) Let $A_n = \{\|X_n - X\| \geq \epsilon\}$. The result follows from Lemma 1.4, Lemma 1.5(i), and Proposition 1.1(iii).
(vi) $X_n \to_p X$ means $\lim_{n\to\infty} P(\|X_n - X\| > \epsilon) = 0$ for every $\epsilon > 0$.
That is, for every $\epsilon > 0$, $P(\|X_n - X\| > \epsilon) < \epsilon$ for $n > n_\epsilon$ (n_ϵ is an integer depending on ϵ).
For every $j = 1, 2, ...$, there is a positive integer n_j such that

$$P(\|X_{n_j} - X\| > 2^{-j}) < 2^{-j}.$$

For any $\epsilon > 0$, there is a k_ϵ such that for $j \geq k_\epsilon$, $P(\|X_{n_j} - X\| > \epsilon) < P(\|X_{n_j} - X\| > 2^{-j})$.
Since $\sum_{j=1}^\infty 2^{-j} = 1$, it follows from the result in (v) that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$.
(vii) The proof for this part is left as an exercise.
(viii) First, by part (iv), we may assume that $X_n \to_{a.s.} X$ (why?).
Proof of (2) implies (1)
Note that (2) (the uniform integrability of $\{\|X_n\|_r^r\}$) implies that $\sup_n E\|X_n\|_r^r < \infty$ (why?)
By Fatou's lemma (Theorem 1.1(i)), $E\|X\|_r^r \leq \liminf_n E\|X_n\|_r^r < \infty$.
Hence, (1) follows if we can show that

$$\limsup_n E\|X_n\|_r^r \leq E\|X\|_r^r. \tag{3}$$

For any $\epsilon > 0$ and $t > 0$, let $A_n = \{\|X_n - X\|_r \leq \epsilon\}$ and $B_n = \{\|X_n\|_r > t\}$. Then

$$\begin{aligned} E\|X_n\|_r^r &= E(\|X_n\|_r^r I_{A_n^c \cap B_n}) + E(\|X_n\|_r^r I_{A_n^c \cap B_n^c}) + E(\|X_n\|_r^r I_{A_n}) \\ &\leq E(\|X_n\|_r^r I_{B_n}) + t^r P(A_n^c) + E\|X_n I_{A_n}\|_r^r. \end{aligned}$$

For $r \leq 1$, $\|X_n I_{A_n}\|_r^r \leq (\|X_n - X\|_r^r + \|X\|_r^r) I_{A_n}$ and

$$E\|X_n I_{A_n}\|_r^r \leq E[(\|X_n - X\|_r^r + \|X\|_r^r) I_{A_n}] \leq \epsilon^r + E\|X\|_r^r.$$

For $r > 1$, an application of Minkowski's inequality leads to

$$\begin{aligned} E\|X_n I_{A_n}\|_r^r &= E\|(X_n - X)I_{A_n} + XI_{A_n}\|_r^r \\ &\le E\left[\|(X_n - X)I_{A_n}\|_r + \|XI_{A_n}\|_r\right]^r \\ &\le \left\{[E\|(X_n - X)I_{A_n}\|_r^r]^{1/r} + [E\|XI_{A_n}\|_r^r]^{1/r}\right\}^r \\ &\le \left\{\epsilon + [E\|X\|_r^r]^{1/r}\right\}^r . \end{aligned}$$

In any case, since ϵ is arbitrary, $\limsup_n E\|X_n I_{A_n}\|_r^r \le E\|X\|_r^r$. This result and the previously established inequality imply that

$$\begin{aligned} \limsup_n E\|X_n\|_r^r &\le \limsup_n E(\|X_n\|_r^r I_{B_n}) + t^r \lim_{n\to\infty} P(A_n^c) \\ &\quad + \limsup_n E\|X_n I_{A_n}\|_r^r \\ &\le \sup_n E(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}}) + E\|X\|_r^r, \end{aligned}$$

since $P(A_n^c) \to 0$. Since $\{\|X_n\|_r^r\}$ is uniformly integrable, letting $t \to \infty$ we obtain (3).

Proof of (1) impies (2)
Let $\xi_n = \|X_n\|_r^r I_{B_n^c} - \|X\|_r^r I_{B_n^c}$. Then $\xi_n \to_{a.s.} 0$ and $|\xi_n| \le t^r + \|X\|_r^r$, which is integrable. By the dominated convergence theorem, $E\xi_n \to 0$; this and (1) imply that

$$E(\|X_n\|_r^r I_{B_n}) - E(\|X\|_r^r I_{B_n}) \to 0.$$

From the definition of B_n, $B_n \subset \{\|X_n - X\|_r > t/2\} \cup \{\|X\|_r > t/2\}$.
Since $E\|X\|_r^r < \infty$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} E(\|X\|_r^r I_{\{\|X_n - X\|_r > t/2\}}) = 0$$

Hence

$$\limsup_n E(\|X_n\|_r^r I_{B_n}) \le \limsup_n E(\|X\|_r^r I_{B_n}) \le E(\|X\|_r^r I_{\{\|X\|_r > t/2\}}).$$

Letting $t \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{t\to\infty} \limsup_n E(\|X_n\|_r^r I_{B_n}) \le \lim_{t\to\infty} E(\|X\|_r^r I_{\{\|X\|_r > t/2\}}) = 0.$$

This proves (2).