Lecture 3: Integration

Integration is a type of “average”.

Definition 1.4
(a) The integral of a nonnegative simple function φ w.r.t. ν is defined as

$$\int \varphi d\nu = \sum_{i=1}^{k} a_i \nu(A_i).$$

(b) Let f be a nonnegative Borel function and let $\mathcal{S}_f$ be the collection of all nonnegative simple functions satisfying $\varphi(\omega) \leq f(\omega)$ for any $\omega \in \Omega$. The integral of f w.r.t. ν is defined as

$$\int f d\nu = \sup \left\{ \int \varphi d\nu : \ \varphi \in \mathcal{S}_f \right\}.$$

(Hence, for any Borel function $f \geq 0$, there exists a sequence of simple functions $\varphi_1, \varphi_2, ...$ such that $0 \leq \varphi_i \leq f$ for all i and $\lim_{n\to\infty} \int \varphi_n d\nu = \int f d\nu$.)
(c) Let f be a Borel function,

$$f_+(\omega) = \max\{f(\omega), 0\}$$

be the positive part of f, and

$$f_-(\omega) = \max\{-f(\omega), 0\}$$

be the negative part of f. (Note that f_+ and f_- are nonnegative Borel functions, $f(\omega) = f_+(\omega) - f_-(\omega)$, and $|f(\omega)| = f_+(\omega) + f_-(\omega)$.) We say that $\int f d\nu$ exists if and only if at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, in which case

$$\int f d\nu = \int f_+ d\nu - \int f_- d\nu.$$

When both $\int f_+ d\nu$ and $\int f_- d\nu$ are finite, we say that f is integrable. Let A be a measurable set and I_A be its indicator function. The integral of f over A is defined as

$$\int_A f d\nu = \int I_A f d\nu.$$

A Borel function f is integrable if and only if $|f|$ is integrable.

For convenience, we define the integral of a measurable function f from $(\Omega, \mathcal{F}, \nu)$ to $(\bar{\mathcal{R}}, \bar{\mathcal{B}})$, where $\bar{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$, $\bar{\mathcal{B}} = \sigma(\mathcal{B} \cup \{\{\infty\}, \{-\infty\}\})$. Let $A_+ = \{f = \infty\}$ and $A_- = \{f = -\infty\}$. If $\nu(A_+) = 0$, we define $\int f_+ d\nu$ to be $\int I_{A_+^c} f_+ d\nu$; otherwise $\int f_+ d\nu = \infty$. $\int f_- d\nu$ is similarly defined. If at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, then $\int f d\nu = \int f_+ d\nu - \int f_- d\nu$ is well defined.

Notation for integrals
$\int f d\nu = \int_\Omega f d\nu = \int f(\omega) d\nu = \int f(\omega) d\nu(\omega) = \int f(\omega)\nu(d\omega)$.
In probability and statistics, $\int X dP = EX = E(X)$ and is called the *expectation* or *expected value* of X.
If F is the c.d.f. of P on $(\mathcal{R}^k, \mathcal{B}^k)$, $\int f(x) dP = \int f(x) dF(x) = \int f dF$.

Example 1.5. Let Ω be a countable set, $\mathcal{F}$ be all subsets of Ω, and ν be the counting measure For any Borel function f,

$$\int f d\nu = \sum_{\omega \in \Omega} f(\omega).$$

Example 1.6. If $\Omega = \mathcal{R}$ and ν is the Lebesgue measure, then the Lebesgue integral of f over an interval $[a, b]$ is written as $\int_{[a,b]} f(x) dx = \int_a^b f(x) dx$, which agrees with the Riemann integral in calculus when the latter is well defined. However, there are functions for which the Lebesgue integrals are defined but not the Riemann integrals.

Properties

Proposition 1.5 (Linearity of integrals). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f and g be Borel functions.
(i) If $\int f d\nu$ exists and $a \in \mathcal{R}$, then $\int (af) d\nu$ exists and is equal to $a \int f d\nu$.
(ii) If both $\int f d\nu$ and $\int g d\nu$ exist and $\int f d\nu + \int g d\nu$ is well defined, then $\int (f+g) d\nu$ exists and is equal to $\int f d\nu + \int g d\nu$.

A statement holds a.e. ν (or simply a.e.) if it holds for all ω in N^c with $\nu(N) = 0$. If ν is a probability, then a.e. may be replaced by a.s.

Proposition 1.6. Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f and g be Borel.
(i) If $f \le g$ a.e., then $\int f d\nu \le \int g d\nu$, provided that the integrals exist.
(ii) If $f \ge 0$ a.e. and $\int f d\nu = 0$, then $f = 0$ a.e.
Proof. (i) Exercise.
(ii) Let $A = \{f > 0\}$ and $A_n = \{f \ge n^{-1}\}$, $n = 1, 2, ...$ Then $A_n \subset A$ for any n and $\lim_{n\to\infty} A_n = \cup A_n = A$ (why?). By Proposition 1.1(iii), $\lim_{n\to\infty} \nu(A_n) = \nu(A)$. Using part (i) and Proposition 1.5, we obtain that

$$n^{-1}\nu(A_n) = \int n^{-1} I_{A_n} d\nu \le \int f I_{A_n} d\nu \le \int f d\nu = 0$$

for any n. Hence $\nu(A) = 0$ and $f = 0$ a.e.

Consequences:
$|\int f d\nu| \le \int |f| d\nu$
If $f \ge 0$ a.e., then $\int f d\nu \ge 0$
If $f = g$ a.e., then $\int f d\nu = \int g d\nu$.