Lecture 30: UMVUE: the method of conditioning

The 2nd method of deriving a UMVUE is conditioning on a sufficient and complete statistic $T(X)$,
i.e., if $U(X)$ is any unbiased estimator of ϑ, then $E[U(X)|T]$ is the UMVUE of ϑ.
We do not need the distribution of T.
But we need to work out the conditional expectation $E[U(X)|T]$.
From the uniqueness of the UMVUE, it does not matter which $U(X)$ is used.
Thus, we should choose $U(X)$ so as to make the calculation of $E[U(X)|T]$ as easy as possible.

Example 3.3. Let $X_1, ..., X_n$ be i.i.d. from the exponential distribution $E(0, \theta)$.
$F_\theta(x) = (1 - e^{-x/\theta})I_{(0,\infty)}(x)$.
Consider the estimation of $\vartheta = 1 - F_\theta(t)$.
$\bar{X}$ is sufficient and complete for $\theta > 0$.
$I_{(t,\infty)}(X_1)$ is unbiased for ϑ,

$$E[I_{(t,\infty)}(X_1)] = P(X_1 > t) = \vartheta.$$

Hence

$$T(X) = E[I_{(t,\infty)}(X_1)|\bar{X}] = P(X_1 > t|\bar{X})$$

is the UMVUE of ϑ. If the conditional distribution of X_1 given $\bar{X}$ is available, then we can calculate $P(X_1 > t|\bar{X})$ directly.
By Basu's theorem (Theorem 2.4), $X_1/\bar{X}$ and $\bar{X}$ are independent.
By Proposition 1.10(vii),

$$P(X_1 > t|\bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{X}|\bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{x}).$$

To compute this unconditional probability, we need the distribution of

$$X_1 \Big/ \sum_{i=1}^n X_i = X_1 \Big/ \left(X_1 + \sum_{i=2}^n X_i \right).$$

Using the transformation technique discussed in §1.3.1 and the fact that $\sum_{i=2}^n X_i$ is independent of X_1 and has a gamma distribution, we obtain that $X_1/\sum_{i=1}^n X_i$ has the Lebesgue p.d.f. $(n-1)(1-x)^{n-2}I_{(0,1)}(x)$.
Hence

$$P(X_1 > t|\bar{X} = \bar{x}) = (n-1)\int_{t/(n\bar{x})}^1 (1-x)^{n-2}dx = \left(1 - \frac{t}{n\bar{x}}\right)^{n-1}$$

and the UMVUE of ϑ is

$$T(X) = \left(1 - \frac{t}{n\bar{X}}\right)^{n-1}.$$

Example 3.4. Let $X_1, ..., X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with unknown $\mu \in \mathcal{R}$ and $\sigma^2 > 0$.
From Example 2.18, $T = (\bar{X}, S^2)$ is sufficient and complete for $\theta = (\mu, \sigma^2)$;
$\bar{X}$ and $(n-1)S^2/\sigma^2$ are independent;
$\bar{X}$ has the $N(\mu, \sigma^2/n)$ distribution;
S^2 has the chi-square distribution χ^2_{n-1}.
Using the method of solving for h directly, we find that
the UMVUE for μ is $\bar{X}$;
the UMVUE of μ^2 is $\bar{X}^2 - S^2/n$;
the UMVUE for σ^r with $r > 1-n$ is $k_{n-1,r}S^r$, where

$$k_{n,r} = \frac{n^{r/2}\Gamma(n/2)}{2^{r/2}\Gamma\left(\frac{n+r}{2}\right)}$$

and the UMVUE of μ/σ is $k_{n-1,-1}\bar{X}/S$, if $n > 2$.

Suppose that ϑ satisfies $P(X_1 \leq \vartheta) = p$ with a fixed $p \in (0,1)$.
Let Φ be the c.d.f. of the standard normal distribution.
Then $\vartheta = \mu + \sigma\Phi^{-1}(p)$ and its UMVUE is $\bar{X} + k_{n-1,1}S\Phi^{-1}(p)$.

Let c be a fixed constant and $\vartheta = P(X_1 \leq c) = \Phi\left(\frac{c-\mu}{\sigma}\right)$.
We can find the UMVUE of ϑ using the method of conditioning.
Since $I_{(-\infty,c)}(X_1)$ is an unbiased estimator of ϑ, the UMVUE of ϑ is

$$E[I_{(-\infty,c)}(X_1)|T] = P(X_1 \leq c|T).$$

By Basu's theorem, the ancillary statistic $Z(X) = (X_1 - \bar{X})/S$ is independent of $T = (\bar{X}, S^2)$.
Then, by Proposition 1.10(vii),

$$\begin{aligned} P\left(X_1 \leq c|T = (\bar{x}, s^2)\right) &= P\left(Z \leq \frac{c-\bar{X}}{S}\bigg|T = (\bar{x}, s^2)\right) \\ &= P\left(Z \leq \frac{c-\bar{x}}{s}\right). \end{aligned}$$

It can be shown that Z has the Lebesgue p.d.f.

$$f(z) = \frac{\sqrt{n}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}(n-1)\Gamma\left(\frac{n-2}{2}\right)}\left[1 - \frac{nz^2}{(n-1)^2}\right]^{(n/2)-2} I_{(0,(n-1)/\sqrt{n})}(|z|)$$

Hence the UMVUE of ϑ is

$$P(X_1 \leq c|T) = \int_{-(n-1)/\sqrt{n}}^{(c-\bar{X})/S} f(z)dz$$

Suppose that we would like to estimate $\vartheta = \frac{1}{\sigma}\Phi'\left(\frac{c-\mu}{\sigma}\right)$, the Lebesgue p.d.f. of X_1 evaluated at a fixed c, where Φ' is the first-order derivative of Φ.

By the previous result, the conditional p.d.f. of X_1 given $\bar{X} = \bar{x}$ and $S^2 = s^2$ is $s^{-1} f\left(\frac{x-\bar{x}}{s}\right)$.
Let f_T be the joint p.d.f. of $T = (\bar{X}, S^2)$.
Then

$$\vartheta = \int\int \frac{1}{s} f\left(\frac{c-\bar{x}}{s}\right) f_T(t) dt = E\left[\frac{1}{S} f\left(\frac{c-\bar{X}}{S}\right)\right].$$

Hence the UMVUE of ϑ is

$$\frac{1}{S} f\left(\frac{c-\bar{X}}{S}\right).$$

Example. Let $X_1, ..., X_n$ be i.i.d. with Lebesgue p.d.f. $f_\theta(x) = \theta x^{-2} I_{(\theta,\infty)}(x)$, where $\theta > 0$ is unknown.
Suppose that $\vartheta = P(X_1 > t)$ for a constant $t > 0$.
The smallest order statistic $X_{(1)}$ is sufficient and complete for θ.
Hence, the UMVUE of ϑ is

$$\begin{aligned}
P(X_1 > t | X_{(1)}) &= P(X_1 > t | X_{(1)} = x_{(1)}) \\
&= P\left(\frac{X_1}{X_{(1)}} > \frac{t}{X_{(1)}} \Big| X_{(1)} = x_{(1)}\right) \\
&= P\left(\frac{X_1}{X_{(1)}} > \frac{t}{x_{(1)}} \Big| X_{(1)} = x_{(1)}\right) \\
&= P\left(\frac{X_1}{X_{(1)}} > s\right)
\end{aligned}$$

(Basu's theorem), where $s = t/x_{(1)}$.
If $s \leq 1$, this probability is 1.
Consider $s > 1$ and assume $\theta = 1$ in the calculation:

$$\begin{aligned}
P\left(\frac{X_1}{X_{(1)}} > s\right) &= \sum_{i=1}^{n} P\left(\frac{X_1}{X_{(1)}} > s, X_{(1)} = X_i\right) \\
&= \sum_{i=2}^{n} P\left(\frac{X_1}{X_{(1)}} > s, X_{(1)} = X_i\right) \\
&= (n-1) P\left(\frac{X_1}{X_{(1)}} > s, X_{(1)} = X_n\right) \\
&= (n-1) P\left(X_1 > sX_n, X_2 > X_n, ..., X_{n-1} > X_n\right) \\
&= (n-1) \int_{x_1 > sx_n, x_2 > x_n, ..., x_{n-1} > x_n} \prod_{i=1}^{n} \frac{1}{x_i^2} dx_1 \cdots dx_n \\
&= (n-1) \int_1^\infty \left[\int_{sx_n}^\infty \prod_{i=2}^{n-1} \left(\int_{x_n}^\infty \frac{1}{x_i^2} dx_i\right) \frac{1}{x_1^2} dx_1\right] dx_n \\
&= (n-1) \int_1^\infty \frac{1}{s x_n^{n-1}} dx_n \\
&= \frac{(n-1) x_{(1)}}{nt}
\end{aligned}$$

This shows that the UMVUE of $P(X_1 > t)$ is

$$h(X_{(1)}) = \begin{cases} \frac{(n-1)X_{(1)}}{nt} & X_{(1)} < t \\ 1 & X_{(1)} \geq t \end{cases}$$

Another way of showing $h(X_{(1)})$ is the UMVUE.
Note that the Lebesgue p.d.f. of $X_{(1)}$ is

$$\frac{n\theta^n}{x^{n+1}} I_{(\theta,\infty)}(x).$$

If $\theta < t$,

$$\begin{aligned} E[h(X_{(1)})] &= \int_\theta^\infty h(x) \frac{n\theta^n}{x^{n+1}} dx \\ &= \int_\theta^t \frac{(n-1)x}{nt} \frac{n\theta^n}{x^{n+1}} dx + \int_t^\infty \frac{n\theta^n}{x^{n+1}} dx \\ &= \frac{\theta^n}{t\theta^{n-1}} - \frac{\theta^n}{t^n} + \frac{\theta^n}{t^n} \\ &= \frac{\theta}{t} \\ &= P(X_1 > t). \end{aligned}$$

If $\theta \geq t$, then $P(X_1 > t) = 1$ and $h(X_{(1)}) = 1$ a.s. P_θ since $P(t > X_{(1)}) = 0$.
Hence, for any $\theta > 0$,

$$E[h(X_{(1)}) = P(X_1 > t).$$