Lecture 30: UMVUE: the method of conditioning

The 2nd method of deriving a UMVUE is conditioning on a sufficient and complete statistic $T(X)$, i.e., if $U(X)$ is any unbiased estimator of $\vartheta$, then $E[U(X)|T]$ is the UMVUE of $\vartheta$. We do not need the distribution of $T$. But we need to work out the conditional expectation $E[U(X)|T]$. From the uniqueness of the UMVUE, it does not matter which $U(X)$ is used. Thus, we should choose $U(X)$ so as to make the calculation of $E[U(X)|T]$ as easy as possible.

Example 3.3. Let $X_1, \ldots, X_n$ be i.i.d. from the exponential distribution $E(0, \theta)$. $F_\theta(x) = (1 - e^{-x/\theta})I_{(0,\infty)}(x)$. Consider the estimation of $\vartheta = 1 - F_\theta(t)$. $\bar{X}$ is sufficient and complete for $\theta > 0$. $I_{(t,\infty)}(X_1)$ is unbiased for $\vartheta$, $E[I_{(t,\infty)}(X_1)] = P(X_1 > t) = \vartheta$. Hence

$$T(X) = E[I_{(t,\infty)}(X_1)|\bar{X}] = P(X_1 > t|\bar{X})$$

is the UMVUE of $\vartheta$. If the conditional distribution of $X_1$ given $\bar{X}$ is available, then we can calculate $P(X_1 > t|\bar{X})$ directly. By Basu’s theorem (Theorem 2.4), $X_1/\bar{X}$ and $\bar{X}$ are independent. By Proposition 1.10(vii),

$$P(X_1 > t|\bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{X} | \bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{x}).$$

To compute this unconditional probability, we need the distribution of

$$X_1/\sum_{i=1}^{n} X_i = X_1/\left( X_1 + \sum_{i=2}^{n} X_i \right).$$

Using the transformation technique discussed in §1.3.1 and the fact that $\sum_{i=2}^{n} X_i$ is independent of $X_1$ and has a gamma distribution, we obtain that $X_1/\sum_{i=1}^{n} X_i$ has the Lebesgue p.d.f. $(n-1)(1-x)^{n-2}I_{(0,1)}(x)$. Hence

$$P(X_1 > t|\bar{X} = \bar{x}) = (n-1)\int_{t/(n\bar{x})}^{1} (1-x)^{n-2}dx = \left( 1 - \frac{t}{n\bar{x}} \right)^{n-1}$$

and the UMVUE of $\vartheta$ is

$$T(X) = \left( 1 - \frac{t}{n\bar{X}} \right)^{n-1}.$$
Example 3.4. Let $X_1, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with unknown $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. From Example 2.18, $T = (\bar{X}, S^2)$ is sufficient and complete for $\theta = (\mu, \sigma^2)$; $\bar{X}$ and $(n - 1)S^2/\sigma^2$ are independent; $\bar{X}$ has the $N(\mu, \sigma^2/n)$ distribution; $S^2$ has the chi-square distribution $\chi^2_{n-1}$. Using the method of solving for $h$ directly, we find that the UMVUE for $\mu$ is $\bar{X}$; the UMVUE of $\mu^2$ is $\bar{X}^2 - S^2/n$; the UMVUE for $\sigma^r$ with $r > 1 - n$ is $k_{n,r}^{n-1}S^r$, where $k_{n,r} = \frac{n^{r/2} \Gamma(n/2)}{2^{r/2} \Gamma(n+r/2)}$ and the UMVUE of $\mu/\sigma$ is $k_{n-1,1}X/S$, if $n > 2$.

Suppose that $\vartheta$ satisfies $P(X_1 \leq \vartheta) = p$ with a fixed $p \in (0, 1)$. Let $\Phi$ be the c.d.f. of the standard normal distribution. Then $\vartheta = \mu + \sigma \Phi^{-1}(p)$ and its UMVUE is $\bar{X} + k_{n-1,1}S\Phi^{-1}(p)$.

Let $c$ be a fixed constant and $\vartheta = P(X_1 \leq c) = \Phi \left( \frac{c-\mu}{\sigma} \right)$. We can find the UMVUE of $\vartheta$ using the method of conditioning. Since $I_{(-\infty,c)}(X_1)$ is an unbiased estimator of $\vartheta$, the UMVUE of $\vartheta$ is

$$E[I_{(-\infty,c)}(X_1)|T] = P(X_1 \leq c|T).$$

By Basu’s theorem, the ancillary statistic $Z(X) = (X_1 - \bar{X})/S$ is independent of $T = (\bar{X}, S^2)$. Then, by Proposition 1.10(vii),

$$P \left( X_1 \leq c | T = (\bar{x}, s^2) \right) = P \left( Z \leq \frac{c - \bar{X}}{S} \bigg| T = (\bar{x}, s^2) \right)$$

$$= P \left( Z \leq \frac{c - \bar{x}}{s} \right).$$

It can be shown that $Z$ has the Lebesgue p.d.f.

$$f(z) = \frac{\sqrt{n} \Gamma \left( \frac{n-1}{2} \right)}{\sqrt{\pi} (n-1) \Gamma \left( \frac{n-2}{2} \right)} \left[ 1 - \frac{n z^2}{(n-1)^2} \right]^{(n/2)-2} I_{(0,(n-1)/\sqrt{n})}(|z|)$$

Hence the UMVUE of $\vartheta$ is

$$P(X_1 \leq c|T) = \int_{-\infty}^{(c-\bar{X})/S} f(z) dz$$

Suppose that we would like to estimate $\vartheta = \frac{1}{\sigma} \Phi' \left( \frac{c-\mu}{\sigma} \right)$, the Lebesgue p.d.f. of $X_1$ evaluated at a fixed $c$, where $\Phi'$ is the first-order derivative of $\Phi$. 

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By the previous result, the conditional p.d.f. of \( X_1 \) given \( \bar{X} = \bar{x} \) and \( S^2 = s^2 \) is 
\[ s^{-1} f \left( \frac{x - \bar{x}}{s} \right) \]

Let \( f_T \) be the joint p.d.f. of \( T = (\bar{X}, S^2) \).

Then
\[ \vartheta = \int \int \frac{1}{s} f \left( \frac{c - \bar{x}}{s} \right) f_T(t) dt = E \left[ \frac{1}{s} f \left( \frac{c - \bar{X}}{S} \right) \right] \]

Hence the UMVUE of \( \vartheta \) is
\[ \frac{1}{S} f \left( \frac{c - \bar{X}}{S} \right) \]

**Example.** Let \( X_1, \ldots, X_n \) be i.i.d. with Lebesgue p.d.f. \( f_\theta(x) = \theta x^{-2} I(\theta, \infty)(x) \), where \( \theta > 0 \) is unknown.

Suppose that \( \vartheta = P(X_1 > t) \) for a constant \( t > 0 \).

The smallest order statistic \( X_{(1)} \) is sufficient and complete for \( \theta \).

Hence, the UMVUE of \( \vartheta \) is
\[ \frac{1}{S} f \left( \frac{c - \bar{X}}{S} \right) \]

(Basu's theorem), where \( s = t/x_{(1)} \).

If \( s \leq 1 \), this probability is 1.

Consider \( s > 1 \) and assume \( \theta = 1 \) in the calculation:

\[
P \left( \frac{X_1}{X_{(1)}} > s \right) = \sum_{i=1}^{n} P \left( \frac{X_1}{X_{(1)}} > s, X_{(1)} = X_i \right)
= \sum_{i=2}^{n} P \left( \frac{X_1}{X_{(1)}} > s, X_{(1)} = X_i \right)
= (n-1) P \left( \frac{X_1}{X_{(1)}} > s, X_{(1)} = X_n \right)
= (n-1) P (X_1 > sX_n, X_2 > X_n, \ldots, X_n > X_n)
= (n-1) \int_{x_1 > s x_n, x_2 > x_n, \ldots, x_{n-1} > x_{n-1}, x_n} \prod_{i=1}^{n} \frac{1}{x_i^2} dx_1 \cdots dx_n
= (n-1) \int_{1}^{\infty} \left[ \int_{1}^{\infty} \prod_{i=2}^{n-1} \left( \int_{x_i}^{\infty} \frac{1}{x_i} dx_i \right) \frac{1}{x_1} dx_1 \right] dx_n
= (n-1) \int_{1}^{\infty} \frac{1}{s x_{n-1}} dx_n
= (n-1) x_{(1)} \frac{1}{nt} \]

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This shows that the UMVUE of \( P(X_1 > t) \) is

\[
h(X_{(1)}) = \begin{cases} 
\frac{(n-1)X_{(1)}}{nt} & X_{(1)} < t \\
1 & X_{(1)} \geq t
\end{cases}
\]

Another way of showing \( h(X_{(1)}) \) is the UMVUE.

Note that the Lebesgue p.d.f. of \( X_{(1)} \) is

\[
\frac{n\theta^n}{x^{n+1}} I_{(\theta, \infty)}(x).
\]

If \( \theta < t \),

\[
E[h(X_{(1)})] = \int_{\theta}^{\infty} h(x) \frac{n\theta^n}{x^{n+1}} dx = \int_{\theta}^{t} \frac{(n-1)x}{nt} \frac{n\theta^n}{x^{n+1}} dx + \int_{t}^{\infty} \frac{n\theta^n}{x^{n+1}} dx = \frac{\theta^n}{t\theta^{n-1}} - \frac{\theta^n}{t^n} = \frac{\theta}{t} = P(X_1 > t).
\]

If \( \theta \geq t \), then \( P(X_1 > t) = 1 \) and \( h(X_{(1)}) = 1 \) a.s. \( P_{\theta} \) since \( P(t > X_{(1)}) = 0 \).

Hence, for any \( \theta > 0 \),

\[
E[h(X_{(1)})] = P(X_1 > t).
\]