Lecture 4: Convergence theorems, change of variable, and Fubini's theorem

$\{f_n : n = 1, 2, ...\}$: a sequence of Borel functions. Can we exchange the limit and integration, i.e.,

$$\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu?$$

Example 1.7. Consider $(\mathcal{R}, \mathcal{B})$ and the Lebesgue measure. Define $f_n(x) = nI_{[0,n^{-1}]}(x)$, $n = 1, 2, ...$ Then $\lim_{n\to\infty} f_n(x) = 0$ for all x but $x = 0$. Since the Lebesgue measure of a single point set is 0, $\lim_{n\to\infty} f_n(x) = 0$ a.e. and $\int \lim_{n\to\infty} f_n(x)dx = 0$. On the other hand, $\int f_n(x)dx = 1$ for any n and, hence, $\lim_{n\to\infty} \int f_n(x)dx = 1$.

Sufficient conditions

Theorem 1.1. Let $f_1, f_2, ...$ be a sequence of Borel functions on $(\Omega, \mathcal{F}, \nu)$.
(i) (Fatou's lemma). If $f_n \geq 0$, then $\int \liminf_n f_n d\nu \leq \liminf_n \int f_n d\nu$.
(ii) (Dominated convergence theorem). If $\lim_{n\to\infty} f_n = f$ a.e. and there exists an integrable function g such that $|f_n| \leq g$ a.e., then $\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu$.
(iii) (Monotone convergence theorem). If $0 \leq f_1 \leq f_2 \leq \cdots$ and $\lim_{n\to\infty} f_n = f$ a.e., then $\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu$.
Proof. (See the textbook).

Note
(a) To apply each part of the theorem, you need to check the conditions.
(b) If the conditions are not satisfied, you cannot apply the theorem, but it does not imply that you cannot exchange the limit and integration.

Example: Let $f_n(x) = \frac{n}{x+n}$, $x \in \Omega = [0, 1]$, $n = 1, 2, ...$ Then $\lim_n f_n(x) = 1$. To apply the DCT, note that $0 \leq f_n(x) \leq 1$. To apply the MCT, note that $0 \leq f_n(x) \leq f_{n+1}(x)$. Hence, $\lim_n \int f_n(x)dx = \int \lim_n f_n(x)dx = \int dx = 1$.

Example 1.8 (Interchange of differentiation and integration). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and, for any fixed $\theta \in \mathcal{R}$, let $f(\omega, \theta)$ be a Borel function on Ω. Suppose that $\partial f(\omega, \theta)/\partial\theta$ exists a.e. for $\theta \in (a, b) \subset \mathcal{R}$ and that $|\partial f(\omega, \theta)/\partial\theta| \leq g(\omega)$ a.e., where g is an integrable function on Ω. Then, for each $\theta \in (a, b)$, $\partial f(\omega, \theta)/\partial\theta$ is integrable and, by Theorem 1.1(ii),

$$\frac{d}{d\theta} \int f(\omega, \theta)d\nu = \int \frac{\partial f(\omega, \theta)}{\partial\theta} d\nu.$$

Theorem 1.2 (Change of variables). Let f be measurable from $(\Omega, \mathcal{F}, \nu)$ to $(\Lambda, \mathcal{G})$ and g be Borel on $(\Lambda, \mathcal{G})$. Then

$$\int_\Omega g \circ f d\nu = \int_\Lambda g d(\nu \circ f^{-1}),$$

i.e., if either integral exists, then so does the other, and the two are the same.

For Riemann integrals, $\int g(y)dy = \int g(f(x))f'(x)dx$, $y = f(x)$.

For a random variable X on $(\Omega, \mathcal{F}, P)$, $EX = \int_\Omega X dP = \int_\mathcal{R} x dP_X$, $P_X = P \circ X^{-1}$
Let Y be a random vector from Ω to $\mathcal{R}^k$ and g be Borel from $\mathcal{R}^k$ to $\mathcal{R}$.

$Eg(Y) = \int_{\mathcal{R}} x dP_{g(Y)} = \int_{\mathcal{R}^k} g(y) dP_Y$
Example: $Y = (X_1, X_2)$ and $g(Y) = X_1 + X_2$.
$E(X_1 + X_2) = EX_1 + EX_2$ (why?) $= \int_{\mathcal{R}} x dP_{X_1} + \int_{\mathcal{R}} x dP_{X_2}$.
We need to handle two integrals involving P_{X_1} and P_{X_2}. On the other hand,
$E(X_1+X_2) = \int_{\mathcal{R}} x dP_{X_1+X_2}$, which involves one integral w.r.t. $P_{X_1+X_2}$. Unless we have some knowledge about the joint c.d.f. of (X_1, X_2), it is not easy to obtain $P_{X_1+X_2}$.

Iterated integration on a product space

Theorem 1.3 (Fubini's theorem). Let ν_i be a σ-finite measure on $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, and let f be a Borel function on $\prod_{i=1}^2 (\Omega_i, \mathcal{F}_i)$ whose integral w.r.t. $\nu_1 \times \nu_2$ exists. Then

$$g(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1$$

exists a.e. ν_2 and defines a Borel function on Ω_2 whose integral w.r.t. ν_2 exists, and

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\nu_1 \times \nu_2 = \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1 \right] d\nu_2.$$

Note: If $f \geq 0$, then $\int f d\nu_1 \times \nu_2$ always exists. Extensions to $\prod_{i=1}^k (\Omega_i, \mathcal{F}_i)$ is straightforward.

Fubini's theorem is *very useful* in
(1) evaluating multi-dimensional integrals (exchanging the order of integrals);
(2) proving a function is measurable;
(3) proving some results by relating a one dimensional integral to a multi-dimensional integral

Example: Exercise 47
Let X and Y be random variables such that the joint c.d.f. of (X, Y) is $F_X(x)F_Y(y)$, where F_X and F_Y are marginal c.d.f.'s. Let $Z = X + Y$. Show that

$$F_Z(z) = \int F_Y(z - x) dF_X(x).$$

Note that

$$\begin{aligned} F_Z(z) &= \int_{x+y \leq z} dF_X(x) dF_Y(y) \\ &= \int \left(\int_{y \leq z-x} dF_Y(y) \right) dF_X(x) \\ &= \int F_Y(z - x) dF_X(x), \end{aligned}$$

where the second equality follows from Fubini's theorem.

Example 1.9. Let $\Omega_1 = \Omega_2 = \{0, 1, 2, ...\}$, and $\nu_1 = \nu_2$ be the counting measure. A function f on $\Omega_1 \times \Omega_2$ defines a double sequence. If $\int f d\nu_1 \times \nu_2$ exists, then

$$\int f d\nu_1 \times \nu_2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f(i, j) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} f(i, j)$$

(by Theorem 1.3 and Example 1.5). Thus, a double series can be summed in either order, if it is well defined.

Proof of Fubini's theorem