Lecture 6: p.d.f. and transformation

Example 1.12. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ whose c.d.f. F_X has a Lebesgue p.d.f. f_X and $F_X(c) < 1$, where c is a fixed constant. Let $Y = \min\{X, c\}$, i.e., Y is the smaller of X and c. Note that $Y^{-1}((-\infty, x]) = \Omega$ if $x \geq c$ and $Y^{-1}((-\infty, x]) = X^{-1}((\infty, x])$ if $x < c$. Hence Y is a random variable and the c.d.f. of Y is

$$F_Y(x) = \begin{cases} 1 & x \geq c \\ F_X(x) & x < c. \end{cases}$$

This c.d.f. is discontinuous at c, since $F_X(c) < 1$. Thus, it does not have a Lebesgue p.d.f. It is not discrete either. Does P_Y, the probability measure corresponding to F_Y, have a p.d.f. w.r.t. some measure? Define a probability measure on $(\mathcal{R}, \mathcal{B})$, called *point mass* at c, by

$$\delta_c(A) = \begin{cases} 1 & c \in A \\ 0 & c \notin A, \end{cases} \qquad A \in \mathcal{B}$$

Then $P_Y \ll m + \delta_c$, where m is the Lebesgue measure, and the p.d.f. of P_Y is

$$\frac{dP_Y}{d(m + \delta_c)}(x) = \begin{cases} 0 & x > c \\ 1 - F_X(c) & x = c \\ f_X(x) & x < c. \end{cases}$$

Example 1.14. Let X be a random variable with c.d.f. F_X and Lebesgue p.d.f. f_X, and let $Y = X^2$. Since $Y^{-1}((-\infty, x])$ is empty if $x < 0$ and equals $Y^{-1}([0, x]) = X^{-1}([-\sqrt{x}, \sqrt{x}\,])$ if $x \geq 0$, the c.d.f. of Y is

$$\begin{aligned} F_Y(x) &= P \circ Y^{-1}((-\infty, x]) \\ &= P \circ X^{-1}([-\sqrt{x}, \sqrt{x}\,]) \\ &= F_X(\sqrt{x}) - F_X(-\sqrt{x}) \end{aligned}$$

if $x \geq 0$ and $F_Y(x) = 0$ if $x < 0$. Clearly, the Lebesgue p.d.f. of F_Y is

$$f_Y(x) = \frac{1}{2\sqrt{x}}[f_X(\sqrt{x}) + f_X(-\sqrt{x})]I_{(0,\infty)}(x).$$

In particular, if

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

which is the Lebesgue p.d.f. of the standard normal distribution $N(0, 1)$, then

$$f_Y(x) = \frac{1}{\sqrt{2\pi x}}e^{-x/2}I_{(0,\infty)}(x),$$

which is the Lebesgue p.d.f. for the chi-square distribution χ_1^2 (Table 1.2). This is actually an important result in statistics.

Proposition 1.8. Let X be a random k-vector with a Lebesgue p.d.f. f_X and let $Y = g(X)$, where g is a Borel function from $(\mathcal{R}^k, \mathcal{B}^k)$ to $(\mathcal{R}^k, \mathcal{B}^k)$. Let $A_1, ..., A_m$ be disjoint sets in $\mathcal{B}^k$ such that $\mathcal{R}^k - (A_1 \cup \cdots \cup A_m)$ has Lebesgue measure 0 and g on A_j is one-to-one with a nonvanishing Jacobian, i.e., the determinant $\mathrm{Det}(\partial g(x)/\partial x) \neq 0$ on A_j, $j = 1, ..., m$. Then Y has the following Lebesgue p.d.f.:

$$f_Y(x) = \sum_{j=1}^{m} \left|\mathrm{Det}\left(\partial h_j(x)/\partial x\right)\right| f_X\left(h_j(x)\right),$$

where h_j is the inverse function of g on A_j, $j = 1, ..., m$.

In Example 1.14, $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$, $g(x) = x^2$, $h_1(x) = -\sqrt{x}$, $h_2(x) = \sqrt{x}$, and $|dh_j(x)/dx| = 1/(2\sqrt{x})$.

Example 1.15. Let $X = (X_1, X_2)$ be a random 2-vector having a joint Lebesgue p.d.f. f_X. Consider first the transformation $g(x) = (x_1, x_1 + x_2)$. Using Proposition 1.8, one can show that the joint p.d.f. of $g(X)$ is

$$f_{g(X)}(x_1, y) = f_X(x_1, y - x_1),$$

where $y = x_1 + x_2$ (note that the Jacobian equals 1). The marginal p.d.f. of $Y = X_1 + X_2$ is then

$$f_Y(y) = \int f_X(x_1, y - x_1) dx_1.$$

In particular, if X_1 and X_2 are independent, then

$$f_Y(y) = \int f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1.$$

Next, consider the transformation $h(x_1, x_2) = (x_1/x_2, x_2)$, assuming that $X_2 \neq 0$ a.s. Using Proposition 1.8, one can show that the joint p.d.f. of $h(X)$ is

$$f_{h(X)}(z, x_2) = |x_2| f_X(z x_2, x_2),$$

where $z = x_1/x_2$. The marginal p.d.f. of $Z = X_1/X_2$ is

$$f_Z(z) = \int |x_2| f_X(z x_2, x_2) dx_2.$$

In particular, if X_1 and X_2 are independent, then

$$f_Z(z) = \int |x_2| f_{X_1}(z x_2) f_{X_2}(x_2) dx_2.$$

Example 1.16 (t-distribution and F-distribution). Let X_1 and X_2 be independent random variables having the chi-square distributions $\chi^2_{n_1}$ and $\chi^2_{n_2}$ (Table 1.2), respectively. The p.d.f.

of $Z = X_1/X_2$ is

$$\begin{aligned} f_Z(z) &= \frac{z^{n_1/2-1} I_{(0,\infty)}(z)}{2^{(n_1+n_2)/2}\Gamma(n_1/2)\Gamma(n_2/2)} \int_0^\infty x_2^{(n_1+n_2)/2-1} e^{-(1+z)x_2/2} dx_2 \\ &= \frac{\Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{z^{n_1/2-1}}{(1+z)^{(n_1+n_2)/2}} I_{(0,\infty)}(z) \end{aligned}$$

Using Proposition 1.8, one can show that the p.d.f. of $Y = (X_1/n_1)/(X_2/n_2) = (n_2/n_1)Z$ is the p.d.f. of the F-distribution F_{n_1,n_2} given in Table 1.2.

Let U_1 be a random variable having the standard normal distribution $N(0,1)$ and U_2 a random variable having the chi-square distribution χ^2_n. Using the same argument, one can show that if U_1 and U_2 are independent, then the distribution of $T = U_1/\sqrt{U_2/n}$ is the t-distribution t_n given in Table 1.2.

Noncentral chi-square distribution

Let $X_1, ..., X_n$ be independent random variables and $X_i = N(\mu_i, \sigma^2)$, $i = 1, ..., n$. The distribution of $Y = (X_1^2 + \cdots + X_n^2)/\sigma^2$ is called the *noncentral chi-square* distribution and denoted by $\chi^2_n(\delta)$, where $\delta = (\mu_1^2 + \cdots + \mu_n^2)/\sigma^2$ is the noncentrality parameter.
$\chi^2_k(\delta)$ with $\delta = 0$ is called a *central* chi-square distribution.
It can be shown (exercise) that Y has the following Lebesgue p.d.f.:

$$e^{-\delta/2} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} f_{2j+n}(x)$$

where $f_k(x)$ is the Lebesgue p.d.f. of the chi-square distribution χ^2_k.
If $Y_1, ..., Y_k$ are independent random variables and Y_i has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i)$, $i = 1, ..., k$, then $Y = Y_1 + \cdots + Y_k$ has the noncentral chi-square distribution $\chi^2_{n_1+\cdots+n_k}(\delta_1 + \cdots + \delta_k)$.

Noncentral t-distribution and F-distribution (in discussion)

Theorem 1.5. (Cochran's theorem). Suppose that $X = N_n(\mu, I_n)$ and

$$X^\tau X = X^\tau A_1 X + \cdots + X^\tau A_k X,$$

where I_n is the $n \times n$ identity matrix and A_i is an $n \times n$ symmetric matrix with rank n_i, $i = 1, ..., k$. A necessary and sufficient condition that $X^\tau A_i X$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i)$, $i = 1, ..., k$, and $X^\tau A_i X$'s are independent is $n = n_1 + \cdots + n_k$, in which case $\delta_i = \mu^\tau A_i \mu$ and $\delta_1 + \cdots + \delta_k = \mu^\tau \mu$.