Topic List for Exam 1

From 311

- **Expected value, Expectation** or **mean value** of X : $E(X) = \mu_X$
- **Variance** of X : $V(X) = \sigma_X^2 = E(X - \mu_X)^2 = E(X^2) - E(X)^2$
- **Standard deviation**(SD) of X is σ_X
- For any constant a and b, $E(aX + b) = aE(X) + b$
- For any constant a and b, $V(aX + b) = a^2 V(X)$

Popular distribution (Ch. 3 and Ch. 4)

- Binomial $Bin(n, p)$: $E(X) = np, V(X) = np(1 - p)$
- Poisson $Poi(\lambda)$: $E(X) = V(X) = \lambda$
- Negative Binomial $nb(r, p)$: $E(X) = \frac{r(1-p)}{p}, V(X) = \frac{r(1-p)}{p^2}$
- Normal Distribution $N(\mu, \sigma^2)$: $E(X) = \mu, V(X) = \sigma^2$

Concepts

- A **point estimate** of a parameter θ is a single number that can be regarded as a sensible value for θ.
- A **point estimator** of θ is used to obtain a point estimate.
  - **Sample Mean**: The sum of a set of n measurements X_1, X_2, $\cdots$, X_n divided by n. Usually denoted by $\bar{X}$.

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{\sum(X_i)}{n}$$

  - **Sample Median**: The middle value when the measurements are arranged from smallest to largest.
    1. when n is odd:
    2. when n is even:
  - **Sample Variance of n observations :** $s^2 = \frac{\sum(X_i - \bar{X})^2}{n-1}$
  - **Sample Standard Deviation :** $s = \sqrt{s^2}$

More on Random Sample

The random variables $X_1, X_2, \ldots, X_n$ are (simple) **random sample** of size n if

1. The X_is are independent(from each other) rv's
2. Every X_i has the same probability distribution.
   (a) $E(X_1) = E(X_2) = \ldots = V(X_n) = \mu_X$
   (b) $V(X_1) = V(X_2) = \ldots = V(X_n) = \sigma_X^2$
3. Denote $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x;\theta)$.
4. Most of time in this course, we only need to deal with random sample.

Unbiased Estimators

- Sample mean $\bar{X}$. If $X_1, X_2, \ldots, X_n$ is random sample from a distribution with mean μ and variance σ^2, then
  1. $E(\bar{X}) = \mu$ (unbiased)
  2. $V(\bar{X}) = \frac{\sigma^2}{n}$
- Sample variance S^2. If $X_1, X_2, \ldots, X_n$ is random sample from a distribution with mean μ and variance σ^2, then

$$\widehat{\sigma^2} = S^2 = \frac{\sum(X_i - \bar{X})^2}{n-1}$$

  is an unbiased estimator of σ^2. Namely $E(S^2) = \sigma^2$
- If we want to decide if an estimator $\widehat{\theta}$ is unbiased, then we need to see if

$$E(\widehat{\theta}) = \theta.$$

The Method of Moments

- **Moment :** Let $X_1, X_2, \ldots X_n \overset{iid}{\sim}$ from a pmf or pdf $f(x;\theta)$. For $k = 1, 2, \ldots$, **the kth population moment**, or **kth moment of the distribution** $f(x;\theta)$ is $E(X^k)$. The kth sample moment is $\frac{1}{n}\sum_{i=1}^{n} X_i^k$.
- **Moment Estimator :** Let $X_1, X_2, \ldots X_n \overset{iid}{\sim} f(x;\theta_1, \ldots, \theta_m)$, where $\theta_1, \ldots, \theta_m$ are unknown parameters. Then the **moment estimators** $\widehat{\theta}_1, \ldots \widehat{\theta}_m$ are obtained by equating the first m sample moment to the corresponding first m population moments and solving for $\theta_1, \ldots, \theta_m$.
- When we use the method of moment, we only use the sample moments to find an estimator of θ.

Maximum Likelihood Estimation

- **Likelihood Function :** Let $X_1, X_2, \ldots X_n$ have joint pmf or pdf $f(x_1, x_2, \ldots, x_n; \theta)$, where θ is unknown parameter. When $x_1, x_2, \ldots, x_n$ are the observed sample values, we compute the probability density associated with our observed data using $f(x_1, x_2, \ldots, x_n; \theta)$. As a function of θ with $x_1, \ldots, x_n$ fixed, this is the **likelihood function**

$$L(\theta|x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n; \theta)$$

- **Maximum Likelihood Estimator :** **Maximum likelihood estimator (MLE)** $\widehat{\theta}$ is the value of θ that maximizes $L(\theta|x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n; \theta)$.

- How do we find MLE?

  1. Write down the likelihood function $L(\theta|x)$ using the given pdf or pmf. This is usually in the form of :
  $$L(\theta|x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta).$$
  2. Take the log on the likelihood to have log likelihood $l(\theta)$, a sum of $\log f(x_i; \theta)$. This is to simplify the calculation.
  3. Differentiate $l(\theta)$ with respect to θ and set the derivative equal to zero. Find the MLE $\widehat{\theta}$ by solving for the root.
  4. Check the second derivative to check if it is the maximum.

- Once we found the MLE $\widehat{\theta}$, then any function of $h(\theta)$ can be obtained by $h(\widehat{\theta})$. e.g., if we have the MLE of σ^2 as $\widehat{\sigma^2}$, then MLE of σ is $\sqrt{\widehat{\sigma^2}}$.