6.4

1. (a) Let $X$ be the number of children weighing at most 2500 grams; then $X \sim \text{Bin}(102, 0.15)$.

Therefore, $P(X \leq 13) = \binom{102}{13} \times 0.15^{13} \times 0.85^{102-x} = 0.3178$.  

Note: Run `pbinom(13, 102, 0.15)`.

(b) Recall that, via § 4.2, 

$X \sim \text{Bin}(n, \pi) \approx N(\mu, \sigma)$, with $\mu = n\pi$ and $\sigma = \sqrt{n\pi(1-\pi)}$.

Thus, 

\[ P(X \leq 13) \approx P\left( Z \leq \frac{X - n\pi + 0.5}{\sqrt{n\pi(1-\pi)}} \right) = P\left( Z \leq \frac{13 - (102 \times 0.15) + 0.5}{\sqrt{102 \times 0.15 \times 0.85}} \right) = P(Z \leq -0.5) = 0.3088. \]

Via § 6.1.3, $\hat{\pi} = \frac{X}{n} \approx N(\pi, \sqrt{\frac{\pi(1-\pi)}{n}})$. The sample-based estimate of $\pi$ is $\hat{\pi} = \frac{13}{102} = 0.128$.

(c) Therefore, 95% margin of error for confidence interval = $(z_{0.025}) \sqrt{\frac{13 \times 89}{102 \times 102}} = 1.96(0.033) = 0.065$. Hence the 95% confidence interval = $(0.128 - 0.065, 0.128 + 0.065) = (0.063, 0.192)$.

The associated $p$-value = $2 P\left( \frac{13}{102} - 0.15 + \frac{0.5}{102} \leq Z \leq \frac{13}{102} + 0.15 + \frac{0.5}{102} \right) = 2 P(Z \leq -0.5) = 0.6177$.

Note that (except for the factor of 2), this is the identical probability computation as in (b)!

Clearly, the 95% confidence interval does indeed contain the hypothesized proportion 0.15, and the $p$-value $>> .05$, so the null hypothesis cannot be rejected at the $\alpha = .05$ level. Thus, our sample data seem to be consistent with the null hypothesis that $\pi = 0.15$. 
(d) □  **R code**

```r
binom.test(13, 102, 0.15)
```

```
Exact binomial test
data:  13 and 102
number of successes = 13, number of trials = 102, p-value = 0.6767
alternative hypothesis: true probability of success is not equal to 0.15
95 percent confidence interval:
  0.06964188 0.20808057
sample estimates:
  probability of success
       0.1274510
```

Note that the answers to (a) and (b) differ a bit, since R utilizes the *exact* binomial distribution in its calculations, whereas we used the normal *approximation* to the binomial.
2. “Smart pill”

(a) \( p\text{-value} = 2 P(\bar{X} \geq 109.9) = 2 P(Z \geq \frac{109.9 - 100}{27/\sqrt{36}}) = 2 P(Z \geq \frac{9.9}{4.5}) = 2 P(Z \geq 2.2) = .0278 \)

(b) C.I. = \((109.9 - 4.5 z_{0.02}, 109.9 + 4.5 z_{0.02})\), where \( z_{0.02} = 1.645, 1.960, 2.575 \), respectively.

<table>
<thead>
<tr>
<th>Significance Level ( \alpha )</th>
<th>Confidence Level 1 - ( \alpha )</th>
<th>Confidence Interval</th>
<th>Decision about ( H_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.10</td>
<td>.90</td>
<td>(102.5, 117.3)</td>
<td>Strong Rejection</td>
</tr>
<tr>
<td>.05</td>
<td>.95</td>
<td>(101.1, 118.7)</td>
<td>Moderate Rejection</td>
</tr>
<tr>
<td>.01</td>
<td>.99</td>
<td>(98.3, 121.5)</td>
<td>Cannot Reject</td>
</tr>
</tbody>
</table>

(c) Generally, a lower significance level \( \alpha \) at the outset of an experiment will make it subsequently more difficult to reject a given null hypothesis, resulting in a more conservative testing procedure. The same \( p\text{-value} \) (e.g., .0278) that results in rejection at some significance level (e.g., \( \alpha = .10 \) or .05) might not result in rejection at a lower level (e.g., \( \alpha = .01 \)). Likewise, a lower significance level gives rise to a larger confidence level, and hence a larger confidence interval, which then may indeed contain the null value (e.g., \( \mu = 100 \)), resulting in an inability to reject. We will see this idea again in “Bonferroni correction.”

3. Serum cholesterol levels

(a) \( H_0: \mu = 211 \text{ mg/dL}, \text{ versus } H_A: \mu \neq 211 \text{ mg/dL} \)

(b) 95% margin of error = \((z_{0.025})(46 / \sqrt{12}) = (1.96)(13.28 \text{ mg/dL}) = 26.03 \text{ mg/dL}.\)

Hence the 95% confidence interval = \((217 - 26.03, 217 + 26.03) = (190.97, 243.03)\).

(c) \( p\text{-value} = 2 P(\bar{X} \geq 217) = 2 P\left(Z \geq \frac{217 - 211}{13.28}\right) = 2 P(Z \geq 0.45) = .653 \).

(d) The 95% confidence interval does indeed contain the hypothesized mean of 211 mg/dL, and the \( p\text{-value} > .05 \) as well, so the null hypothesis cannot be rejected at the \( \alpha = .05 \) level. Interpretation: Based on the data from this sample, there has been no statistically significant difference demonstrated between the true mean serum cholesterol level of the population of hypertensive male smokers, and the mean serum cholesterol level of the general male population (211 mg/dL).

(e) The 95% acceptance region for \( H_0 \) is \((217 - 26.03, 217 + 26.03) = (184.97, 237.03)\). Therefore, the complementary 95% rejection region is \((-\infty, 184.97] \cup [237.03, +\infty)\). This is entirely consistent with our conclusion in (d), as the sample mean of 217 is in the former region.
4. Plasma aluminum levels

(a) \( H_0: \mu = 4.13 \ \mu g/L, \) versus \( H_A: \mu \neq 4.13 \ \mu g/L. \)

(b) 95\% margin of error = \((t_{9,.025}) (7.13 / \sqrt{10}) = (2.262) (2.255 \ \mu g/L) = 5.1 \ \mu g/L. \) Hence the 95\% confidence interval = \((37.2 - 5.1, 37.2 + 5.1) = (32.1, 42.3) \ \mu g/L. \)

(c) \( p\)-value = \(2 \cdot P(\bar{X} \geq 37.20) = 2 \cdot P(T_9 \geq \frac{37.20 - 4.13}{2.255}) = 2 \cdot P(T_9 \geq 14.67) \ll 0.05. \)

\( \square \) Run \( 2 \cdot \text{pt}(14.67, \text{df} = 9, \text{lower.tail} = \text{F}) \) for the exact \( p\)-value.

(d) The 95\% confidence interval certainly does not contain the hypothesized mean of 4.13 \( \mu g/L, \) and the \( p\)-value \( < 0.05 \) as well, so the null hypothesis is strongly rejected at the \( \alpha = .05 \) level. \textbf{Interpretation:} Based on the data from this sample (37.2 \( \mu g/L), \) there is an extremely statistically significant difference demonstrated between the true mean plasma aluminum level of the population of infants receiving antacids, and the mean plasma aluminum level of the population of infants not receiving antacids (4.13 \( \mu g/L). \)

(e) \( H_0: \mu \leq 4.13 \ \mu g/L, \) versus \( H_A: \mu > 4.13 \ \mu g/L. \) The \( p\)-value for this one-sided test is \( P(\bar{X} \geq 37.20) = P(T_9 \geq 14.67), \) or exactly half the \( p\)-value for the two-sided test above, consequently, \( < 0.025 \) \( < 0.05. \) As before, this leads to an extremely strong rejection of the null hypothesis in favor of the alternative, at the \( \alpha = .05 \) significance level. \textbf{Interpretation:} Based on the data from this sample, the true mean plasma aluminum level of the population of infants receiving antacids is significantly higher than the mean plasma aluminum level of the population of infants not receiving antacids (4.13 \( \mu g/L). \)

4.4 / 2 - Money wheel

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( f_i )</th>
<th>( p(x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>13</td>
<td>0.1625</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>0.2000</td>
</tr>
<tr>
<td>30</td>
<td>19</td>
<td>0.2375</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>0.4000</td>
</tr>
</tbody>
</table>

\( n = 80 \)

\( \bar{x} = (10)(0.1625) + (20)(0.2) + (30)(0.2375) + (40)(0.4) = $28.75 \)


\( \mu = (10)(0.25) + (20)(0.25) + (30)(0.25) + (40)(0.25) = $25 \)

\( \sigma^2 = (10 - 25)^2 (0.25) + (20 - 25)^2 (0.25) + (30 - 25)^2 (0.25) + (40 - 25)^2 (0.25) = 125.00 \) \( \therefore \sigma = $11.18 \) \( \checkmark \)
5. Money wheel – revisited (see above)

(a) With \( n = 80 \), \( \bar{x} = 28.75 \), \( \mu = 25 \), and \( s^2 = 125 \) (i.e., \( s = \sqrt{125} \)), we have \( \bar{X} \approx N(25, \frac{\sqrt{125}}{\sqrt{80}}) \)
\[ = N(25, 1.25) \]. Therefore...

\[ p\text{-value} = 2P(\bar{X} \geq 28.75) = 2P(Z \geq \frac{28.75 - 25}{1.25}) = 2P(Z \geq 3) = 2(0.00135) = \boxed{0.0027 < .05}, \]
so the null hypothesis \( H_0: \mu = 25 \) can be rejected at the .05 significance level. However, this null hypothesis is true, because as the calculations in 4.4/2 show, the mean \( \mu \) is indeed equal to 25! Therefore, our decision to reject is incorrect, and we have committed a Type I error, the probability of which is the significance level \( \alpha = .05 \), by definition. In other words, the mean \( \bar{x} = 28.75 \) of this particular random sample is one of the unlucky ones that led to an erroneous conclusion, an event that should only occur by chance 5% of the time, if \( H_0 \) is true.

(b) Now testing false null hypothesis \( \mu_0 = 28 \), vs. \( \mu_1 = 25 \), using the same sample data as above.

\[ \text{Power} = 1 - \beta = P(Z \leq -z_{.025} + \Delta\sqrt{n}) = P\left(Z \leq -1.96 + \frac{|25 - 28|}{\sqrt{125}}\right) = P(Z \leq 0.44) = 0.67003, \]
i.e., only 67% power of correctly rejecting \( H_0: \mu = 28 \) in favor of the alternative \( H_A: \mu = 25 \). Indeed, \( p\text{-value} = 2P(\bar{X} \geq 28.75) = 2P\left(Z \geq \frac{28.75 - 28}{1.25}\right) = 2P(Z \geq 0.6) = 2(0.27425) = \boxed{0.54851 > .05 = \alpha}, \]
which indicates that this false null hypothesis cannot be rejected at this significance level. This is a Type II error.

8. We have power \( = 1 - \beta = P(Z \leq z) \), where \( z = -1.96 + 0.5 \sqrt{n} \) for \( n = 25, 16, 9, 4, 1 \), respectively. Hence \( z = 0.54, 0.04, -0.46, -0.96, -1.46 \), from which it follows via the Z-table [or via the R command \texttt{pnorm(c(0.54, 0.04, -0.46, -0.96, -1.46))}] that power (customarily expressed as a percentage) = \text{70.5\%, 51.6\%, 32.3\%, 16.9\%, 7.2\%}, respectively. Clearly, the power steadily decreases as sample size decreases.