1. The different types of **random variable** on a population.

![Diagram of random variable types]

2. Basic Definitions

   - How to represent the **sample space** $S$ of all experimental outcomes via a Venn Diagram, and two or more events $(E, F, \ldots)$ as subsets of $S$.

   **Example**: Experiment = “Randomly pick a single playing card from a standard deck (and replace).”
   
   $S = \{A\spadesuit, A\heartsuit, A\clubsuit, A\diamondsuit, 2\spadesuit, \ldots, 10\spadesuit, 10\heartsuit, 10\clubsuit, 10\diamondsuit, J\spadesuit, J\heartsuit, J\clubsuit, J\diamondsuit, Q\spadesuit, Q\heartsuit, Q\clubsuit, Q\diamondsuit, K\spadesuit, K\heartsuit, K\clubsuit, K\diamondsuit\}$
   
   $A =$ “Pick an Ace” = $\{A\spadesuit, A\heartsuit, A\clubsuit, A\diamondsuit\}$
   
   $B =$ “Pick a Black card” = $\{A\spadesuit, 2\spadesuit, \ldots, 10\spadesuit, J\spadesuit, 10\spadesuit, \ldots, K\spadesuit\}$
   
   $C =$ “Pick a Clubs card” = $\{A\clubsuit, 2\clubsuit, \ldots, 10\clubsuit\}$
   
   $D =$ “Pick a Diamonds card” = $\{A\diamondsuit, 2\diamondsuit, \ldots\}$

3. Basic Definition and Properties of Probability (for any two events $E$ and $F$)

   - The general notion of **probability** $P(E)$ of an event $E$ as the “limiting value” of its “long-run” relative frequency $\frac{\text{# times } E \text{ occurs}}{\text{# experimental trials}}$, as the experimental trials are repeated indefinitely.

   - $0 \leq P(E) \leq 1$

   - $P(E) = \frac{\text{# outcomes in } E}{\text{# outcomes in } S} \ldots \text{ONLY IF the outcomes are equally likely}$

   **Example (cont’d)**: $P(A) = 4/52$ and $P(B) = 26/52$… **IF** the deck is “fair,” i.e., $P(\text{each card}) = 1/52$.

   - **Complement** $E^c = “\text{Not } E” \quad P(E^c) = 1 - P(E) \quad \text{“Complement Rule”}$
• Intersection $E \cap F = \{E \text{ and } F\}$

Example (cont’d): $A \cap B = \{\text{Ace, Ace}\}$, so that $P(A \cap B) = 2/52$.

Special Case: $E$ and $F$ are disjoint or mutually exclusive if $E \cap F = \emptyset$, i.e., $P(E \cap F) = 0$.

Example (cont’d): With $C = \text{“Clubs”}$ and $D = \text{“Diamonds”}$ above, $C \cap D = \emptyset$, so that $P(C \cap D) = 0$.

• Union $E \cup F = \{E \text{ or } F\}$

Example (cont’d): $P(A \cup B)$ = 4/52 + 26/52 - 2/52 = 28/52

“Inclusion-Exclusion” (for $\geq 2$ events)

• DeMorgan’s Laws: $(E \cup F)^c = E^c \cap F^c (E \cap F)^c = E^c \cup F^c$

• Distributive Laws: $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$

How to construct and use a $2 \times 2$ probability table for two events $E$ and $F$:

<table>
<thead>
<tr>
<th>Events</th>
<th>$E$</th>
<th>$E^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$P(E \cap F)$</td>
<td>$P(E^c \cap F)$</td>
</tr>
<tr>
<td>$F^c$</td>
<td>$P(E \cap F^c)$</td>
<td>$P(E^c \cap F^c)$</td>
</tr>
</tbody>
</table>

4. Conditional Probability (of any event $E$, given any event $F$)

• $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$, which can be rewritten as $P(E \cap F) = P(E \mid F) \cdot P(F)$ “Multiplication Rule”

This latter formula can be expanded into a full tree diagram, where successive “branch probabilities” are multiplied together to yield intersection probabilities.

• Special Case: $E$ and $F$ are statistically independent if either of the following conditions holds:
  - $P(E \mid F) = P(E)$, ...or likewise, $P(F \mid E) = P(F)$
  - $P(E \cap F) = P(E) \cdot P(F)$ (from above)

Example (cont’d): $P(A \cap B) = 1/26$ is indeed equal to the product of $P(A) = 1/13$ times $P(B) = 1/2$, so events $A = \text{“Pick an Ace”}$ and $B = \text{“Pick a Black card”}$ are statistically independent (… but not disjoint, since $A \cap B = \{\text{Ace, Ace}\}$)!

Example: $E$ and $F$ below are statistically independent because each cell probability is equal to the product of its corresponding row and column marginal probabilities (e.g., $0.28 = 0.7 \times 0.4$, etc.), but events $G$ and $H$ are not, i.e., they are statistically dependent.

<table>
<thead>
<tr>
<th>Events</th>
<th>$E$</th>
<th>$E^c$</th>
<th>$F$</th>
<th>$F^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>0.28</td>
<td>0.42</td>
<td>0.70</td>
<td>0.30</td>
</tr>
<tr>
<td>$E^c$</td>
<td>0.12</td>
<td>0.18</td>
<td>0.30</td>
<td>0.70</td>
</tr>
<tr>
<td>$G$</td>
<td>0.15</td>
<td>0.55</td>
<td>0.70</td>
<td>0.30</td>
</tr>
<tr>
<td>$G^c$</td>
<td>0.25</td>
<td>0.05</td>
<td>0.30</td>
<td>0.70</td>
</tr>
<tr>
<td>$H$</td>
<td>0.40</td>
<td>0.60</td>
<td>1.00</td>
<td>0.40</td>
</tr>
<tr>
<td>$H^c$</td>
<td>0.40</td>
<td>0.60</td>
<td>1.00</td>
<td>0.40</td>
</tr>
</tbody>
</table>
5. **Bayes’ Rule**

![](https://via.placeholder.com/150)

Given:
- Prior Probabilities
- Conditional Probabilities

Then... **Posterior Probabilities** are obtained via the formula:

\[
\frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) P(B_j)}, \quad i = 1, 2, \ldots, n
\]

Finally, compare each prior to its corresponding posterior. **INTERPRET IN CONTEXT!**

**“Law of Total Probability”**

<table>
<thead>
<tr>
<th>B_i</th>
<th>B_2</th>
<th>\ldots</th>
<th>B_n</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{Prior probabilities} \times \text{Conditional probabilities}</td>
<td>\text{P(A)}</td>
<td>\text{P(A)}</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>P(B_1)</td>
<td>P(B_2)</td>
<td>\ldots</td>
</tr>
<tr>
<td>A^c</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>