EXAM # 2

PLEASE SHOW ALL WORK!

<table>
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<tr>
<th>Problem</th>
<th>Points</th>
<th>Grade</th>
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<tr>
<td>1</td>
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1. Two male ladybugs are sitting at opposite corners of a large window composed of individual square glass panes, as shown. At the same instant, they start walking at the same rate along the pane edges, one edge at a time, in the hope of eventually reaching the female sitting at the center of the window. At the start of every “move” along a single edge, each male randomly – and independently – selects only either the horizontal or vertical direction shown, each with the probability indicated. However, the female is only willing to wait for each male to complete a total of \( n = 5 \) horizontal and vertical moves along the edges, before she becomes impatient and flies away. Given this information, answer each of the following.

[Hint: Let the random variable \( X_1 = \) “number of ‘\( \rightarrow \)’ moves in \( n = 5 \) independent moves by Male 1,” and \( X_2 = \) “number of ‘\( \leftarrow \)’ moves in \( n = 5 \) independent moves by Male 2.”]

(a) Calculate the **probability** that Male 1 meets the female in \( n = 5 \) moves. \( \quad \) (5 pts)

(b) Calculate the **probability** that Male 2 meets the female in \( n = 5 \) moves. \( \quad \) (5 pts)

(c) Calculate the **probability** that both males meet the female in \( n = 5 \) moves. \( \quad \) (5 pts)

(d) Calculate the **probability** that neither male meets the female in \( n = 5 \) moves. \( \quad \) (5 pts)

(e) Calculate the **probability** that only Male 1 meets the female in \( n = 5 \) moves. \( \quad \) (5 pts)

(f) Calculate the **probability** that only Male 2 meets the female in \( n = 5 \) moves. \( \quad \) (5 pts)
2. An experiment involving a fair deck of 52 cards is to be conducted. Cards will be randomly selected from the deck with replacement. The random variable of interest is \( X \) = “Number of trials until a ‘face card’ (i.e., Jack, Queen, or King) first appears.”

- Can the resulting outcomes be considered a sequence of Bernoulli trials? Why? (2 pts)

- Which probability distribution is the most appropriate to model this experiment? Determine all parameter values. That is, \( X \sim ? \) (2 pts)

- Calculate the probability that a face card will first appear on the fourth trial. Show all work… A calculator-derived answer alone will not earn any credit. (3 pts)

(b) A second experiment involving the same deck is to be conducted. Cards will be randomly selected from the deck, but without replacement, until a “face card” (i.e., Jack, Queen, or King) first appears.

- Can the resulting outcomes be considered a sequence of Bernoulli trials? Why? (2 pts)

- Calculate the probability that a face card will first appear on the fourth trial. Show all work… A calculator-derived answer alone will not earn any credit. (3 pts)

(c) Suppose that in general, a population of finite size \( N \) units contains \( s \) “Successes.” Units are to randomly drawn from the population without replacement.

- Formally prove that the random variable \( X \) = “Number of trials until the first Success appears” has pmf

\[
p(x) = P(X = x) = \binom{N - x}{s - 1} \binom{N}{s}^{-1} \text{ for } x = 1, 2, 3, 4, \ldots
\]

**Hint:** Start by generalizing the procedure in (b), then apply some algebra. (6 pts)

(d) Use this formula to recalculate the probability in (b), and show agreement in your answers. (2 pts)
3.
(a) Starting from 0, a flea randomly jumps either one unit or three units to the right, with fixed probability $p$ or $1-p$ respectively, as shown in the probability histogram below. Also shown is the accompanying pmf chart, for the random variable $X$ = “Number of units jumped.”

<table>
<thead>
<tr>
<th>$x$</th>
<th>$p(x)$</th>
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<tbody>
<tr>
<td>1</td>
<td>$p$</td>
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<tr>
<td>3</td>
<td>$1-p$</td>
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Calculate the expected number of units jumped, i.e., the mean of $X$, in terms of $p$. Express in simplest possible form. Show all work! (3 pts)

Calculate the variance of $X$, in terms of $p$. Express in simplest possible form. Show all work! (5 pts)

(b) Starting from 0, an ant crawls horizontally, and can randomly stop anywhere ($Y$) in the interval $[0, 4]$, with corresponding probability given by the piecewise uniform pdf $f(y) = \begin{cases} \frac{p}{2}, & 0 \leq y < 2 \\ \frac{1-p}{2}, & 2 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$ (and 0 elsewhere), as illustrated below.*

* As usual, $0 < p < 1$. The illustration shown is for $0 < p < 0.5$ without loss of generality, for if $0.5 < p < 1$, the heights are just reversed.
Calculate the expected distance crawled, i.e., the mean of $Y$, in terms of $p$. Verify that this mean is the same as in (a), and give a brief explanation for why this should be. (It does not have to be mathematically rigorous.) Show all work! (5 pts)

Calculate the variance of $Y$, in terms of $p$. Express in simplest possible form. Show all work! (5 pts)

(c)
Determine the cdf $F(y) = P(Y \leq y)$, for all $y$ in $[0, 4]$, in terms of $p$. Show all work! (10 pts)

Sketch a labeled graph of the cdf below, for $0 < p < 0.5$. (5 pts)

(d) For all $0 < h < 2$, consider the interval $[2-h, 2+h]$, symmetric about $Y = 2$. Using the cdf found in (c), compute the probability $P(2-h \leq Y \leq 2+h)$ in terms of $h$, and confirm that it does not depend on the value of $p$. Show all work! (2 pts)
4. The relation between the continuous variable \( X = \text{Foot Length (inches)} \) and the discrete variable \( Y = \text{Men's Shoe Size} \) is shown in the two horizontal scales below. (For example, men’s shoe size 9 corresponds to foot length between \( 10\frac{2}{6} \) and \( 10\frac{3}{6} \) inches.) Suppose that in a certain population of men, \( X \) is normally distributed with mean \( \mu = 11 \) inches, and standard deviation \( \sigma = 2/3 \) inches, i.e., \( X \sim N(11, 2/3) \), as illustrated.

(a) Calculate the probability that a randomly selected man has a shoe size of 12 or larger. 
*Show all work!* (5 pts)

(b) Calculate the probability that a randomly selected man has a shoe size larger than 12.
*Show all work!* (5 pts)

(c) Calculate the probability that a randomly selected man has a shoe size of exactly 12.
*Show all work!* (2 pts)

(d) Calculate the probability that a randomly selected man has a shoe size of exactly 12, given that his shoe size is 12 or larger. *Show all work!* (3 pts)
Cumulative Probabilities of the Standard Normal Distribution $\Phi(0, 1)$

**Note:** To linearly interpolate for "in-between" values, solve

$$z_{interpolate} = \frac{z_{high} - z_{low}}{\Phi(z_{high}) - \Phi(z_{low})} \cdot (\Phi(z_{interpolate}) - \Phi(z_{low})) + z_{low}$$

for either $\Phi(z_{high})$ or $\Phi(z_{low})$, whichever required, given the other.