Math 340–Problem Solving Seminar, Fall 2001, Problem Set 6

(1) A non-negative integer $f(n)$ is assigned to each positive integer $n$ in such a way that the following conditions are satisfied:

(a) $f(mn) = f(m) + f(n)$, for all positive integers $m, n$;

(b) $f(n) = 0$ whenever the units digit of $n$ (in base 10) is a 3’
    (i.e., $f(3) = 0, f(13) = 0, f(23) = 0, ...$); and

(c) $f(10) = 0$.

Prove that $f(n) = 0$, for all positive integers $n$.

(2) Given any three numbers $a, b$, and $c$ between $0$ and $1$, prove that not all of the expressions $a(1-b)$, $b(1-c)$, and $c(1-a)$ can be greater than $\frac{1}{3}$.

(3) Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where
    $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$.
    Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots.

(4) Given a point $(a, b)$ with $0 > b > a$, determine the minimum perimeter of a triangle with one vertex at $(a, b)$ one on the $x$-axis, and one on the line $y = x$. 