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DIFFUSION SMOOTHING ON BRAIN SURFACE VIA FINITE ELEMENT METHOD

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ABSTRACT

Surface data such as the segmented cortical surface of the human brain plays an important role in medical imaging. To increase the signal-to-noise ratio for data residing on the brain surface, the data is usually diffused. Most of diffusion equation approach for triangulated mesh data is based on the finite element method and a system of linear equations are iteratively solved without the explicit representation of the Laplace-Beltrami operator. Such implicit formulation requires inverting large sparse matrix that is required in most finite element methods.

The novelty of our paper is in the explicit representation of the Laplace-Beltrami operator derived from the finite element method itself. The Laplace-Beltrami operator is expressed as a weighted averaging operator where the weights are expressed in terms of the interior angles and the area of triangles. The weights are computed by actually solving the system of linear equations in the finite element method and inverting a matrix in a computational algebra system MAPLE. Afterwards the diffusion equation is solved via a simple finite difference scheme that speed up computation.

1. INTRODUCTION

When solving a diffusion equation on a curved surface as a way to increase the signal-to-noise ratio (SNR), the Laplacian somehow has to incorporate the geometry of the curved surface. The extension of the Euclidean Laplacian to an arbitrary Riemannian manifold is called the *Laplace-Beltrami operator* [5]. Although diffusion equations are widely used to smooth out data in imaging analysis [7, 8, 9, 10, 11], there is a very few paper that uses diffusion to smooth out data residing on the brain surface [1, 3]. Many previous work involving diffusion are the problem of surface fairing [6, 11] or anisotropic smoothing of intensity images where the image intensities are taken as surfaces to be smoothed [9].

In this paper, we show how to represent the Laplace-Beltrami operator explicitly for an arbitrary triangular surface mesh using the finite element method (FEM). Afterwards the explicit representation of the Laplacian is used in the finite difference method (FDM) for solving a diffusion equation iteratively on the human brain surface for the cortical thickness and curvature measurements [3]. A similar discrete representation that is based on geometric arguments can be found in [6, 12].

2. LAPLACE-BELTRAMI OPERATOR

Suppose we have an orientable smooth twice-differentiable (C^2) 2-dimensional surface S in $\mathbb{R}^3$. Then we have a parameterization of S:

$$X(u) = \{x_1(u), x_2(u), x_3(u) : u = (u^1, u^2) \in D\}$$

for some planar domain D. Let T_pS be a tangent space at any $p = X(u) \in S$ such that partial derivatives

$$X_1(u) = \partial_{u^1} X(u), X_2(u) = \partial_{u^2} X(u)$$

form a basis in T_p. Any vector $d\xi \in T_pS$ can be written as $d\xi = du^1 X_1 + du^2 X_2$ for some constants du^1 and du^2 and the length of the vector $d\xi$ is

$$d\xi^2 \equiv \langle d\xi, d\xi \rangle = \sum_{i,j} g_{ij} du^i du^j, \quad (1)$$

where the inner products $g_{ij} = \langle X_i, X_j \rangle$ are called the *Riemannian metric tensor*. Then the Laplace-Beltrami operator Δ_X corresponding to the surface parameterization X is defined as

$$\Delta_X F = \frac{1}{|g|^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i} \Big(|g|^{1/2} g^{ij} \frac{\partial F}{\partial u^j} \Big). \quad (2)$$

Since the Laplace-Beltrami operator is *self adjoint* with respect to the L^2 norm for any C^2 functions F and G on S, it can be written as

$$\int_S G \Delta F dS = -\int_S \langle \nabla F, \nabla G \rangle \, dS = \int_S F \Delta G dS. \quad (3)$$

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3. FINITE ELEMENT METHOD

Let N_T be the number of triangles in the triangular mesh $\hat{S}$ that is the discrete estimation of true surface S. We seek an approximate solution F_i in triangle T_i such that the solution $F_i(x,t)$ is continuous across neighboring triangles. The approximate solution F for the whole region is then

$$F(x,t) \doteq \sum_{i=1}^{N_T} F_i(x,t). \quad (4)$$

A slightly different formulation of FEM for the surface flattening problem is given in [2]. Let $p_{i_1}, p_{i_2}, p_{i_3}$ be the vertices of element T_i. In T_i, we interpolate F_i linearly by

$$F_i(x,t) = \sum_{j=1}^{3} \xi_{i_j}(x) F(p_{i_j}, t), \quad (5)$$

where nonnegative ξ_{i_k} are given by the *barycentric coordinates* [13]. In the barycentric coordinates, any point $x \in T_i$ is uniquely determined by two conditions:

$$x = \sum_{k=1}^{3} \xi_{i_k}(x) p_{i_k}, \sum_{k=1}^{3} \xi_{i_k}(x) = 1.$$

3.1. Discrete Diffusion Equation

Let G be an arbitrary piecewise linear function given by

$$G(x) = \sum_{i=1}^{N_T} \xi_{i_1}(x) G_{i_1} + \xi_{i_2}(x) G_{i_2} + \xi_{i_3}(x) G_{i_3},$$

where $G_{i_k} = G(p_{i_k})$ are the values of function G evaluated at vertex p_{i_k} of T_i. Another piecewise linear function $F(x,t)$ is given similarly so that $F_{i_k} = F(p_{i_k}, t)$.

Then from equation (3), the integral version of diffusion equation $\partial_t F = \Delta F$ can be written as

$$\int_{T_i} G \partial_t F \; dT = -\int_{T_i} \langle \nabla F, \nabla G \rangle \; dT. \quad (6)$$

The left-hand term in (6) is

$$\begin{aligned} \int_{T_i} G \partial_t F \; dT &= \sum_{k,l=1}^{3} G_{i_k} \partial_t F(p_{i_l}, t) \int_{T_i} \xi_{i_k} \xi_{i_l} \; dT \\ &= [G_i]'[A^i] \frac{d}{dt}[F_i], \end{aligned}$$

where $[G_i] = (G_{i_1}, G_{i_2}, G_{i_3})'$, $[F_i] = (F_{i_1}, F_{i_2}, F_{i_3})'$ and 3×3 matrix $[A^i] = (A^i_{kl}), A^i_{kl} = \int_{T_i} \xi_{i_k} \xi_{i_l} \; dT$. It can be shown that

$$[A^i] = \frac{|T_i|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

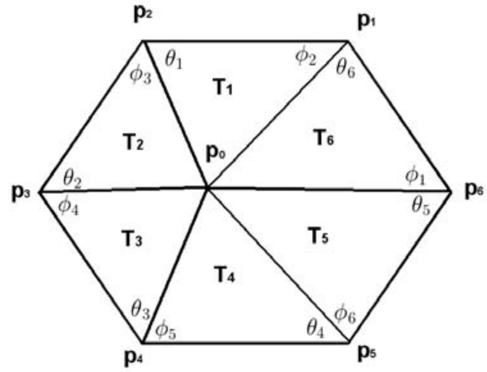

Fig. 1. A typical triangular elements.

where $|T_i|$ is the area of the triangular element T_i [13]. Similarly the right-hand term in (6) can be written as

$$\begin{aligned} \int_{T_i} \langle \nabla F, \nabla G \rangle dT &= \sum_{k,l=1}^{3} G_{i_k} F(p_{i_l}, t) \int_{T_i} \langle \nabla \xi_{i_k}, \nabla \xi_{i_l} \rangle dT \\ &= [G_i]'[C^i][F_i], \end{aligned}$$

where $[C^i] = (C^i_{kl}), C^i_{kl} = \int_{T_i} \langle \nabla \xi_{i_k}, \nabla \xi_{i_l} \rangle \; dT$. Since T_i is planar, the gradient $\nabla \xi_{i_k}$ becomes the standard planar gradient. Following [13, 14], it can be shown that $2[C^i]$ is the matrix of the following form:

$$\begin{pmatrix} \cot\theta_{i_2} + \cot\theta_{i_3} & -\cot\theta_{i_3} & -\cot\theta_{i_2} \\ -\cot\theta_{i_3} & \cot\theta_{i_1} + \cot\theta_{i_3} & -\cot\theta_{i_1} \\ -\cot\theta_{i_2} & -\cot\theta_{i_1} & \cot\theta_{i_1} + \cot\theta_{i_2} \end{pmatrix},$$

where θ_{i_k} is the interior angle of vertex p_{i_k}. Equating the left and right hand sides, we get

$$[G_i]'[A^i] \frac{d}{dt}[F_i] = -[G_i]'[G^i][F_i]. \quad (7)$$

Since the equation (7) should be satisfied for an arbitrary vector $[G_i]$, we have a system of ordinary differential equations (ODE) given by

$$\frac{d[F_i]}{dt} = -[A^i]^{-1}[C^i][F_i] \text{ for all } i. \quad (8)$$

3.2. Assembling Elements

Having discretized an element, the next step is to assemble all such elements in m incident triangles around vertex p. Let $p_1, \cdots, p_m$ be the m neighboring vertices around $p = p_0$ in the counter-clockwise direction. Let p, p_i, p_{i+1} be the vertices of the element T_i (Figure 1). Then from combining elements, we have

$$\begin{aligned} \int_{T_1 \cup \cdots \cup T_m} \langle \nabla F, \nabla G \rangle \; dT &= \sum_{i=1}^{m} \int_{T_i} \langle \nabla F, \nabla G \rangle \; dT \\ &= [G]'[C][F], \end{aligned}$$

where $[F] = [F(p,t), F(p_1,t), \cdots, F(p_m,t)]'$ and $[G] = [G(p), G(p_1), \cdots, G(p_m)]'$. The matrix $[C] = (C_{ij})$ is called the *global coefficient matrix*, which is the assemblage of individual element coefficients. The contribution to C_{ij} comes from all elements containing vertices i and j. In the case of a hexagonal triangulation in Figure 1, $[C]$ is given by

$$\begin{pmatrix} C_{00}^1+\cdots+C_{00}^6 & C_{01}^1+C_{01}^6 & C_{02}^1+C_{02}^2 & C_{03}^2+C_{03}^3 & C_{04}^3+C_{04}^4 & C_{05}^4+C_{05}^5 & C_{06}^5+C_{06}^6 \\ C_{01}^1+C_{01}^6 & C_{11}^1+C_{11}^6 & C_{12}^1 & 0 & 0 & 0 & C_{16}^6 \\ C_{02}^1+C_{02}^2 & C_{12}^1 & C_{22}^1+C_{22}^2 & C_{23}^2 & 0 & 0 & 0 \\ C_{03}^2+C_{03}^3 & 0 & C_{23}^2 & C_{33}^2+C_{33}^3 & C_{34}^3 & 0 & 0 \\ C_{04}^3+C_{04}^4 & 0 & 0 & C_{34}^3 & C_{44}^3+C_{44}^4 & C_{45}^4 & 0 \\ C_{05}^4+C_{05}^5 & 0 & 0 & 0 & C_{45}^4 & C_{55}^4+C_{55}^5 & C_{56}^5 \\ C_{06}^5+C_{06}^6 & C_{16}^6 & 0 & 0 & 0 & C_{56}^5 & C_{66}^5+C_{66}^6 \end{pmatrix}.$$

Similarly

$$\int_{T_1\cup\cdots\cup T_m} G\frac{\partial F}{\partial t}\,dT = [G]'[A]\frac{d[F]}{dt},$$

where $[A] = (A_{ij})$ has the same structure as $[C]$, i.e. write $A_{01} = A_{01}^1 + A_{01}^6$ instead of $C_{01} = C_{01}^1 + C_{01}^6$. Equating the above equations we have

$$[G]'[A]\frac{d[F]}{dt} = -[G]'[C][F]. \quad (9)$$

Since equation (9) should be satisfied for an arbitrary piecewise linear function G, we have a discrete diffusion equation on m elements $T_1, \cdots, T_m$ given by

$$\frac{d[F]}{dt} = -[A]^{-1}[C][F]. \quad (10)$$

At first glance, it seems like we need to solve a system of linear equations iteratively. Note that the first row of the simultaneous ODE (10) gives the discrete diffusion equation at the vertex $p = p_0$:

$$\frac{dF(p,t)}{dt} = -\sum_{i,k=0}^{m} A_{0k}^{-1}C_{ki}F(p_i,t), \quad (11)$$

where A_{0k}^{-1} is the $0k$-th element of A^{-1}. Comparing this with the diffusion equation $\partial_t F(p,t) = \Delta F(p,t)$, we can see that the right-hand side of equation (11) is the discrete estimation of Laplacian of function F evaluated at vertex p. Simplifying the matrix computation using the computational algebraic system `MAPLE`, we have the FEM estimation for the Laplace-Beltrami operator given by

$$\widehat{\Delta}F(p) = \sum_{i=1}^{m} w_i\big(F(p_i) - F(p)\big) \quad (12)$$

with the weights $w_i = (\cot\theta_i + \cot\phi_i)/|T|$, where θ_i and ϕ_i are the two angles opposite to the edge $p_i - p$ and $|T| = \sum_{i=1}^m |T_i|$ is the sum of the areas of the incident triangles (Figure 1).

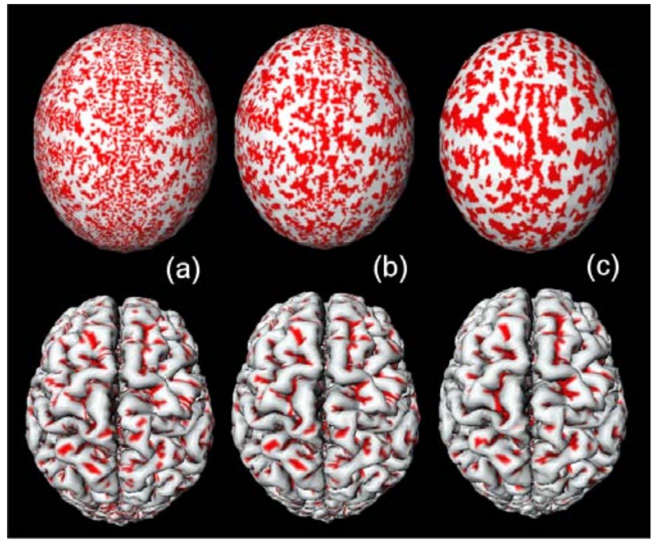

Fig. 2. Diffusion smoothing was applied to smooth out the mean curvature of the brain cortex and projected onto a sphere to show how hidden sulcal pattern can be enhanced over time. (a) initial mean curvature. (b) after 20 iterations with $\delta t = 0.2$. (c) after 100 iterations we are beginning to see much clear sulcal pattern.

4. FINITE DIFFERENCE METHOD

The diffusion equation is solved by the finite difference scheme:

$$F(p, t_{n+1}) = F(p, t_n) + (t_{n+1} - t_n)\widehat{\Delta}F(p, t_n) \quad (13)$$

where $\widehat{\Delta}F(p,t_n)$ is estimated by (12). We may fix the iteration step size $t_{n+1} - t_n = \delta t$. For the convergence of the finite difference scheme, δt is chosen to satisfy the harmonic condition

$$\min_i F(p_i, t_n) \le F(p, t_n + \delta t) \le \max_i F(p_i, t_n). \quad (14)$$

For more detailed convergence condition, see [4].

Note that the Laplace-Beltrami operator in the conformal coordinate system [5] (u^1, u^2) can be written as

$$\Delta = \frac{\partial^2}{\partial(u^1)^2} + \frac{\partial^2}{\partial(u^2)^2}.$$

So we can define the FWHM of diffusion smoothing locally as the FWHM of the corresponding Gaussian kernel in the conformal coordinate system. Then diffusion smoothing with N iterations and the step size δt would be equivalent to Gaussian kernel smoothing with

$$\mathrm{FWHM} = 4(\ln 2)^{1/2}\sqrt{N\delta t}.$$

Computing the linear weights for the Laplace-Beltrami operator takes a fair amount of time in `MATLAB` but once the

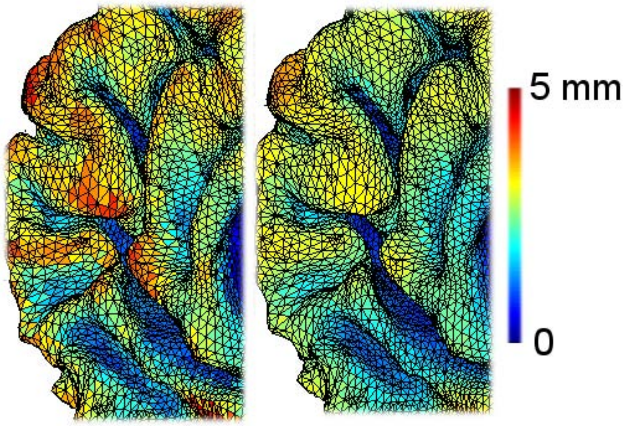

Fig. 3. Left: cortical thickness computed at the posterior right hemisphere of the autistic brain. Right: Diffusion smoothing estimation of the cortical thickness metric. We can see huge noise reduction. Gyri have thicker gray matter compared to thinner sulci.

weights are computed, it is applied through the whole iteration repeatedly and the actual finite difference scheme takes less than one minute for 100 iterations in Pentium 4.

5. CONCLUSIONS

Based on the FEM, we discretized a diffusion equation in a triangular mesh patch centered around a vertex and solved a system of equation using a computational algebraic system `MAPLE`. It turns out that the Laplace-Beltrami operator can be represented as a weighted averaging operation where the weights are given in terms of the interior angles and the areas of triangles. Based on the FEM estimate for the Laplace-Beltrami operator, we iteratively run the finite difference with a temporal step size that satisfies a convergence criterion. This diffusion smoothing would be highly useful in smoothing fMRI data [1] and anatomical data [3] that are residing on the fixed brain surface and would be useful in statistical inference based on the random fields theory.

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