Diffusion Smoothing on the Cortical Surface

M.K. Chung[1], K.J. Worsley[12], J. Taylor[1],
J.O. Ramsay[3], S. Robbins[2], A.C. Evans[2]
[1]Department of Mathematics and Statistics,
[3]Department of Psychology,
[2]Montreal Neurological Institute,
McGill University, Montreal, Canada

Summary

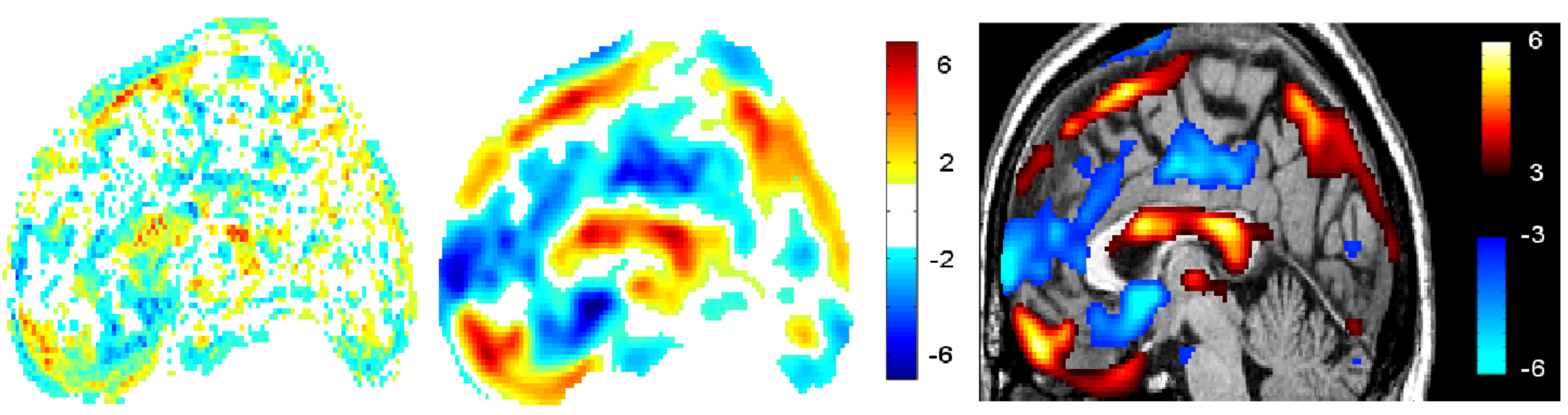
Gaussian kernel smoothing has been widely used in smoothing 2D or 3D brain images. It does not work on curved surfaces such as the human brain surfaces. By reformulating Gaussian kernel smoothing as a solution to a diffusion equation on a Riemannian manifold, the smoothing method can be generalized to an arbitrary curved surface. This generalization is called *diffusion smoothing* and it has been used in the analysis of fMRI data on the brain surface (Andrade *et al.*, 2001) and detecting the regions of local surface area change in brain development (Chung, 2001).

Gaussian Kernel Smoothing

An integral version of isotropic Gaussian kernel smoothing of the function ***f(x)*** is defined as the convolution of the Gaussian kernel ***K(x)*** with the signal ***f(x)***:

$$F^*(\mathbf{x}, h) = \int_{\Re^n} K\Big(\frac{\mathbf{x}-\mathbf{y}}{h}\Big)\frac{f(\mathbf{y})}{h^n}\, d\mathbf{y}$$

$$K(\mathbf{x}) = (2\pi)^{-n/2}\exp(-\|\mathbf{x}\|^2/2)$$



t-map of brain tissue growth and loss in children before and after Gaussian kernel smoothing with 10mm FWHM on the mid-sagittal section (Chung *et al*, 2001). The most rapid tissue growth appears in the splenium and the isthmus of the corpus callosum. Gaussian kernel smoothing increases the signal-to-noise ratio (SNR).

Diffusion Smoothing

It can be shown that the convoluted signal $F(\mathbf{x}, t) = F^*(\mathbf{x}, \sqrt{2t}\,)$ is the solution of a diffusion equation

$$\frac{\partial F}{\partial t} = \Delta F,\ F(\mathbf{x}, 0) = f(\mathbf{x})$$

where the n-dimensional Laplacian is given by $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. The amount of smoothing is determined by the full width at the half maximum (FWHM) of Gaussian kernel

$$\mathrm{FWHM} = 4(\ln 2)^{1/2}\sqrt{t} = 2(2\ln 2)^{1/2}h$$

Since the cortical surface in non-Euclidean, the above Laplacian is not well defined on the cortical surface. The generalization of the Laplacian to an arbitrary curved surface is called the *Laplace-Beltrami operator* and it is defined in terms of the Riemannian metric tensors. For the Riemannian metric $ds^2 = \sum_{i,j=1}^n g_{ij}\ du^i du^j$, the Laplace-Beltrami operator is given by

$$\Delta F = \frac{1}{|g|^{1/2}}\sum_{i,j=1}^n \frac{\partial}{\partial u^i}\Big(|g|^{1/2} g^{ij}\frac{\partial F}{\partial u^j}\Big)$$

where $g^{-1} = (g^{ij})$ and $|g| = \det(g_{ij})$.

Finite Element Method

The ASP algorithm (MacDonald, *et al.*, 2001) is used to extract the outer cortical surfaces each consisting of 81,920 triangles from MR scans. At this surface sampling rate, the average intervertex distance is 3-4 mm. In order to estimate the Laplace-Beltrami operator on a triangulated cortical surface, we use the *finite element method* (FEM) (Chung, 2001). Let $F(\mathbf{p}_i)$ be the signal on the *i*-th node $\mathbf{p}_i$ in the triangulation. If $\mathbf{p}_1,\ldots,\mathbf{p}_m$ are *m*-neighboring nodes around $\mathbf{p}=\mathbf{p}_0$, the Laplace-Beltrami operator at **p** is estimated by

$$\widehat{\Delta F}(\mathbf{p}) = \sum_{i=1}^m w_i(F(\mathbf{p}_i) - F(\mathbf{p}))$$

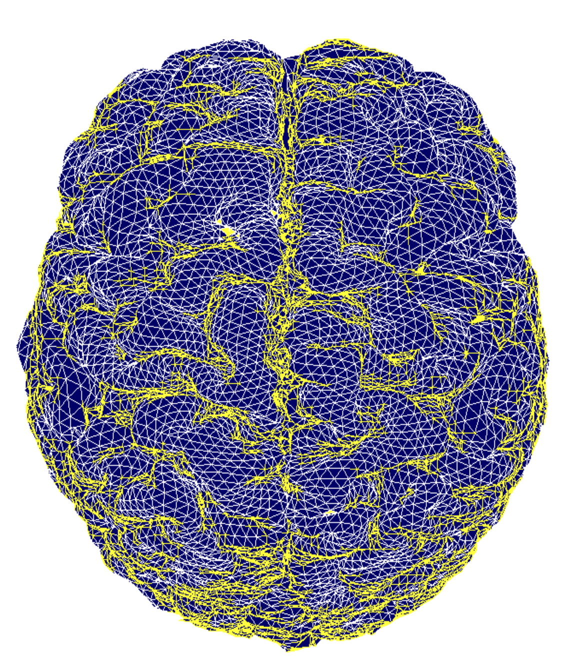
with the weights

$$w_i = (\cot\theta_i + \cot\phi_i)/|T|$$

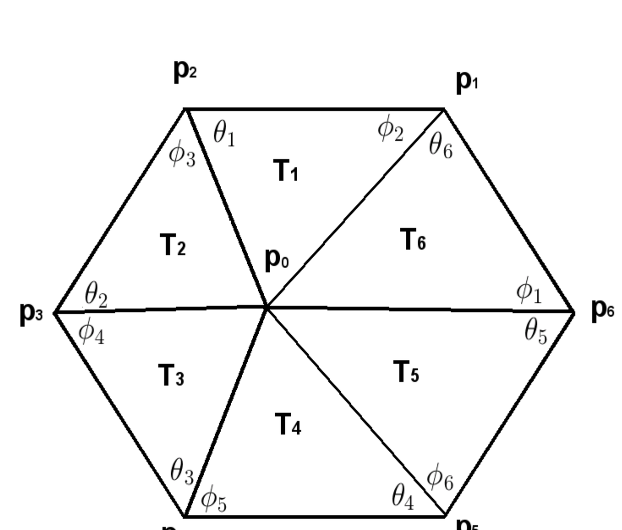
where θ_i and ϕ_i are the two angles opposite to the edge $\mathbf{p}_i - \mathbf{p}$ in triangles and $|T|$ is the sum of the areas of *m*-incident triangles at **p**. Then the diffusion equation is solved via the *finite difference scheme*:

$$F(\mathbf{p}, t_{n+1}) = F(\mathbf{p}, t_n) + (t_{n+1} - t_n)\widehat{\Delta F}(\mathbf{p}, t_n)$$

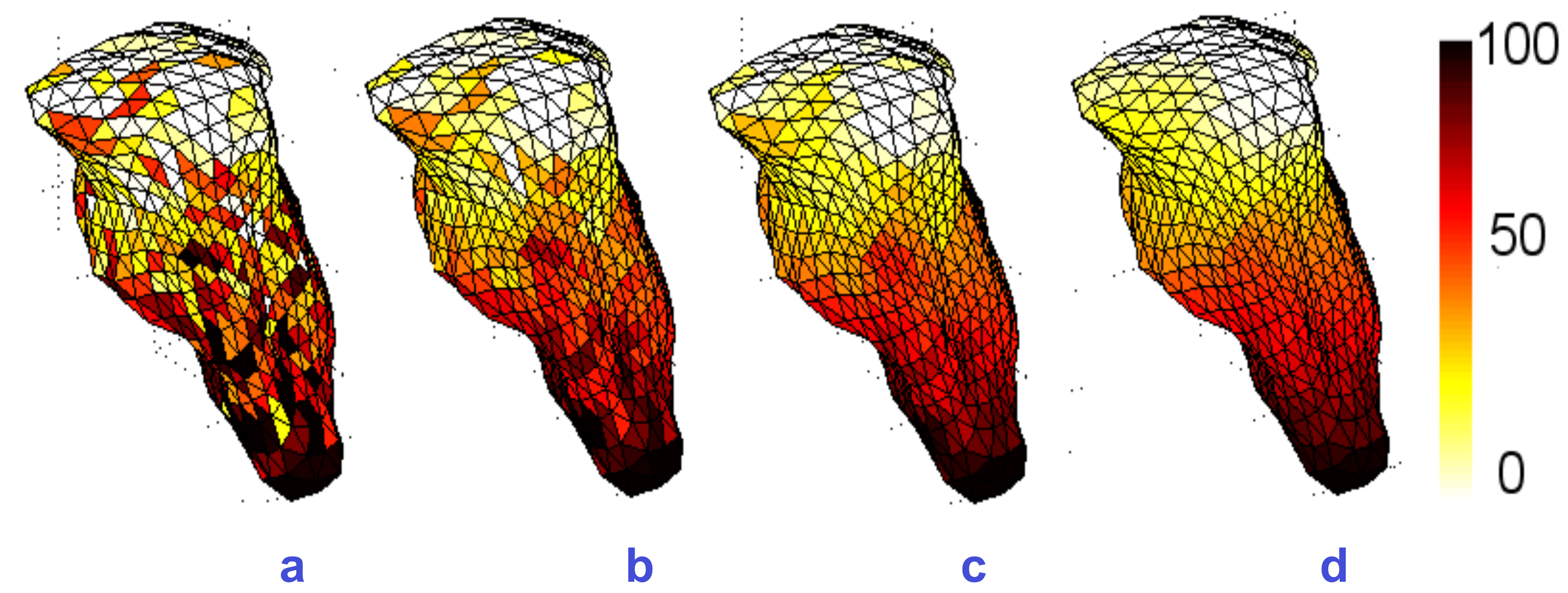
with the initial condition $F(\mathbf{p}_i, t_0) = f(\mathbf{p}_i)$. After *N*-iterations, the diffused signal is locally equivalent to Gaussian kernel smoothing with FWHM = $4(\ln 2)^{1/2}N^{1/2}(t_N - t_0)^{1/2}$.



A typical triangular mesh of the outer cortical surface consisting of 81,920 triangles and 40,962 vertices.



Simulation on Brain Stem Surface



Diffusion smoothing of an artificial heat distribution on the triangulated mesh of the brain stem consisting of 1280 triangles. The artificial signal was generated with Gaussian noise to illustrate how the finite difference scheme works with different iteration step sizes. **a.** The initial heat distribution. **b.** After 10 iterations with the iteration step size 0.5. **c.** After 20 iterations. **d.** After 50 iterations with the iteration step size 0.2, If the iteration step size is larger than a certain critical value, the iteration will breaks down.

Example: Smoothing Mean Curvature

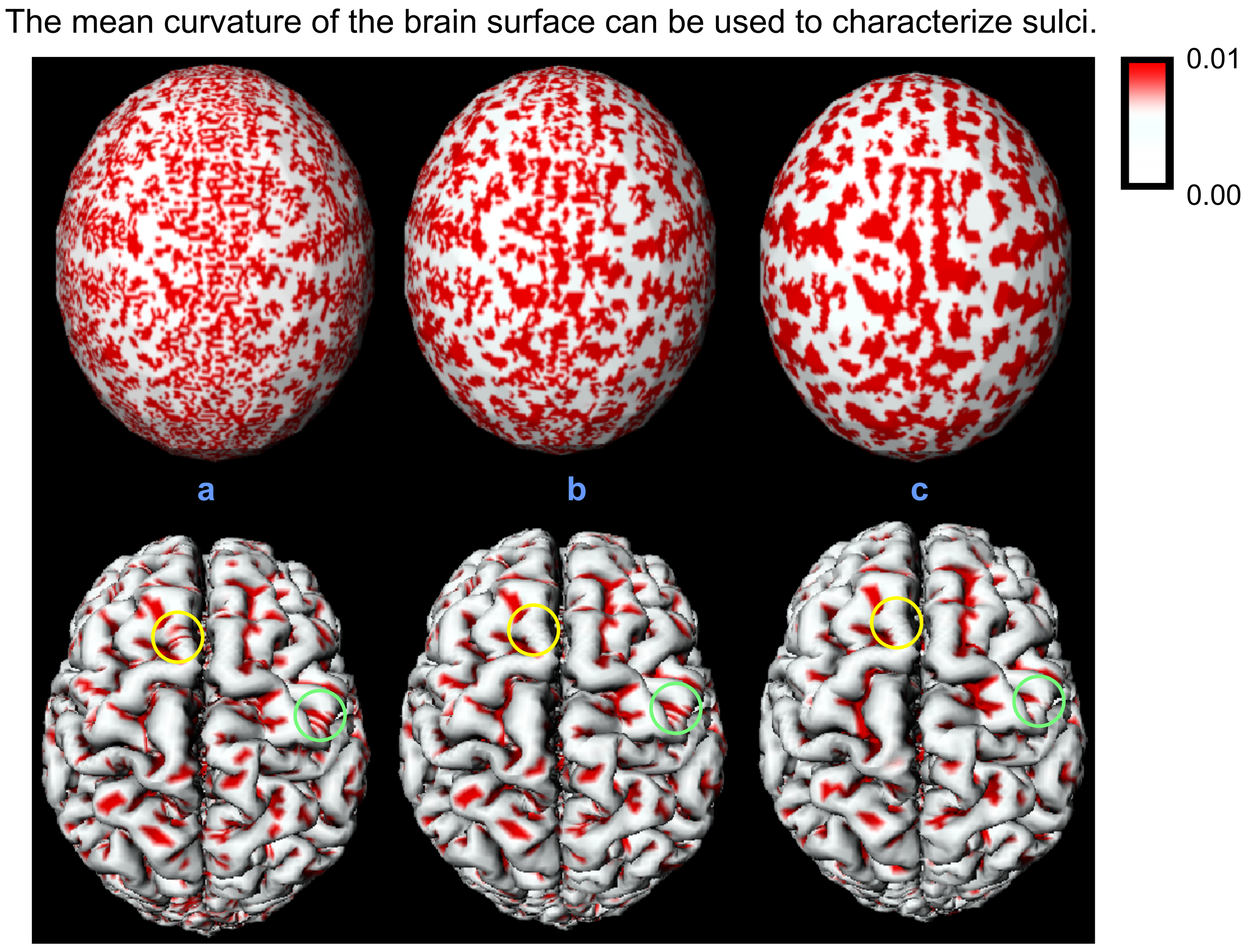
The cortical surface can be locally parameterized by a quadratic polynomial

$$z(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \frac{1}{2}\beta_3 x_1^2 + \beta_4 x_1 x_2 + \frac{1}{2}\beta_5 x_2^2$$

The unknown coefficients are estimated by the least-squares estimation. The mean curvature of the cortical surface is given by

$$K_M = \frac{\beta_3(1+\beta_2^2) + \beta_5(1+\beta_1^2) - 2\beta_1\beta_2\beta_4}{(1+\beta_1^2+\beta_2^2)^{3/2}}$$

The mean curvature of the brain surface can be used to characterize sulci.



The figure shows the diffusion smoothing of the mean curvature with 5mm FWHM. The mean curvature of the outer cortex is mapped onto an ellipsoid consisting of 81,920 triangles preserving the connectivity. Note that the diffusion was run directly on the cortical surface and mapped onto the ellipsoid later. **a.** Before the iteration. **b.** After 40 iterations with the iteration step size 0.02. **c.** After 100 iterations. If the smoothing were based on simple inter-nodal averaging, such sulcal pattern can not be obtained.

Discussion

- Gaussian kernel smoothing can be generalized to cortical surfaces enabling surface-based statistical analysis.
- The numerical implementation is freely available as **Matlab** code on the web at http://www.math.mcgill.ca/chung
- Diffusion smoothing presented here can be further generalized to spatially adaptive nonisotropic diffusion smoothing.

References

[1] Andrade, A., Kherif, F., Mangin, J., Worsley, K.J., Paradis, A., Simon, O., Dehaene, S., Le Bihan, D., Poline, J.B., Detection of fMRI Activation on the Cortical Surface. *Human Brain Mapping* , **12**:79-93, 2001.
[2] Chung, M.K., *Statistical Morphometry in Neuroanatomy*, PhD Thesis, McGill University, 2001.
[3] Chung, M.K., Worsley, K.J., Paus, T., Cherif, C., Collins, D.L., Giedd, J.N., Rapoport, J.L., Evans, A.C. A Unified Statistical Approach for Deformation-Based Morphometry, *NeuroImage* **14** (in press).
[4] MacDonald, J.D., Kabani, N., Avis, D., Evans, A.C., Automated 3D Extraction of Inner and Outer Surfaces of Cerebral Cortex from MRI, *NeuroImage* **12**:340-356, 2000.