1 Affine transforms

Anatomical objects extracted from 3D medical images are aligned using affine transformations to remove the global size differences. Figure 2 illustrates an example where one mandible surface is aligned to another via an affine transform.

The affine transform $T$ of point $p = (p_1, \cdots, p_d) \in \mathbb{R}^d$ to $q = (q_1, \cdots, q_d)$ is given by

$$q = Rp + c,$$

where the matrix $R$ corresponds to rotation, scaling and shear while $c$ corresponds to translation. Note that the affine transform is nonlinear. Note that

$$T(ap + bq) = R(ap + bq) + c = (aRp + c) + (bRq + c) - c = aT(p) + bT(q) - c.$$

Unless $c \neq 0$, the affine transform is not linear due to the translation term.

The affine transform can be easily made into a linear form by augmenting the transform. The affine transform can be rewritten in a matrix form as

$$\begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} R & c \\ 0, \cdots 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

(1)

Let

$$A = \begin{pmatrix} R & c \\ 0, \cdots 0 & 1 \end{pmatrix}.$$
Then trivially $A$ is linear on $\begin{pmatrix} p \\ 1 \end{pmatrix}$. The matrix $A$ is the most often used form for affine registration in medical imaging.

The inverse of the affine transform is given by

$$p = R^{-1}q - R^{-1}c.$$ 

This can be written in a matrix form as

$$\begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} R^{-1} & -R^{-1}c \\ 0, \cdots, 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix}. $$

We denote the matrix form of the inverse as

$$A^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}c \\ 0, \cdots, 0 & 1 \end{pmatrix}.$$ 

## 2 Least squares estimation

For $n$ given points $p_i = (p_{i1}, \cdots, p_{id})'$ and its corresponding affine transformed points $q_i = (q_{i1}, \cdots, q_{id})'$, we can estimate the affine transform matrix $A$ in the least squares fashion. For points $p_i$ and $q_i$, we have

$$q_i = Ap_i. $$

(2)
The least squares estimation of $A$ is given by

$$
\hat{A} = \arg \min_{A \in \mathcal{G}} \sum_{i=1}^{n} \| q_i - Ap_i \|^{2},
$$

where $\mathcal{G}$ is an affine group. We rewrite (1) as

$$
\left( \begin{array}{c}
q_1 \\
\vdots \\
q_n \\
\end{array} \right) = \left( \begin{array}{cc}
R & c \\
\end{array} \right) \left( \begin{array}{c}
p_1 \\
\vdots \\
p_n \\
\end{array} \right).
$$

Then the least squares estimation is trivially given as

$$
\left( \begin{array}{c}
\hat{R} \\
\hat{c} \\
\end{array} \right) = Q P^{\prime} (P P^{\prime})^{-1}.
$$

This can be easily implemented in MATLAB. Then the points $p_i$ are mapped to $\hat{R}p_i + \hat{c}$, which may not coincide with $q_i$ in general.

If we try to solve the least squares problem with an additional constraint, the problem can become complicated. For instance, if we restrict $R$ to be rotation only, i.e. $R^{\prime}R = I$, iterative updates of least squares estimation are needed [1].

References

Figure 2: Mandible F155-12-08 (yellow) is used as a template and mandibles F155-12-08 is affinely aligned to F155-12-08 by matching 24 manually identified landmarks. The affine transform does not match the landmarks perfectly but minimizes the distance between them in a least squares fashion.