1. The cumulative distribution of $X$ is defined as $F(x) = P(X \leq x)$. $F(-\infty) = 0$, $F(\infty) = 1$.

2. Problem. For $X \sim \text{Binomial}(n, p)$,

$$F(x) = \sum_{y=0}^{i} \binom{n}{y} p^y (1-p)^{n-y}$$

for integer $i \leq x \leq i + 1$.

3. Problem. For positive integer valued discrete random variable $X$, show

$$\mathbb{E}X = \sum_{x=1}^{\infty} P(X \geq x).$$

Note

$$\sum_{x=1}^{\infty} P(X \geq x) = \sum_{x=1}^{\infty} \sum_{y=x}^{\infty} P(y) = \sum_{y=1}^{\infty} \sum_{x=1}^{y} p(y) = \sum_{y=1}^{\infty} yp(y) = \mathbb{E}X.$$ 

4. The constant $\pi$ is usually computed by summing up infinite series, i.e.

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right) = 3.141592\ldots.$$ 

The convergence for computing this series is very slow. There is a differently different way to compute the digits of $\pi$ based on the theory of probability. Let $A$ be the area of a unit circle bounded by a unit square $B$. Suppose we throw darts at the square at random. Let $(X, Y)$ be the coordinates of the dart. It is a pair of random variables. $X$ is continuously defined at every point of the continuous domain $A$. This type of random variable is called a continuous random variable. Now define binary random variable $Z = 1$ if a dart hits $A$ and $Z = 0$ if it hits $B/A$. Now we define $Z_i$ similarly for the $i$-th dart thrown. Then we can show

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} Z_i}{n} \to \pi.$$ 

The computer code MATLAB to simulate 1,000,000 dart throwing experiment.

```matlab
x=rand(100000,1);
y=rand(100000,1);
rsquare=x.^2+y.^2;
accept=find(rsquare <1);
ans =
    2 3 4 5 6 7 8 10 11 15 ...
plot(x(accept),y(accept),'.')
>>4*size(accept,1)/1000000
ans =
    3.1336
```

5. Formally, $X$ is a continuous random variable if there is nonnegative function $f$, called probability density, such that

$$P(X \in B) = \int_B f(x) \, dx.$$ 

for every $B \subset \mathbb{R}$. Let $B = [a, b]$. Then

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx.$$ 

By letting $a = b$,

$$P(a \leq X \leq a) = \int_a^a f(x) \, dx = 0.$$ 

By letting $a = -\infty$ and $b = \infty$,

$$1 = P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x) \, dx = 0.$$ 

6. Let $X$ have the probability density function $f(x) = c|x|$ for $-1 < x < 1$ and $f(x) = 0$ otherwise, where $|x|$ denotes the absolute value of $x$. Find the value $c$. solution. To find $c$, we integrate

$$\int_{-1}^{1} c|x| \, dx = \int_{-1}^{0} -cx \, dx + \int_{0}^{1} cx \, dx = 1.$$ 

So $c = 1$. 

```matlab
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