Nov 22, 2005

1. The covariance between $X$ and $Y$ is
   \[ \text{Cov}(X, Y) = \mathbb{E}(X - \mu_X)(Y - \mu_Y), \]
   where $\mu_X = \mathbb{E}X, \mu_Y = \mathbb{E}Y$. It measures the relationship between two random variables.

2. **Correlation coefficient** is given by
   \[ \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}. \]
   It is the normalized version of the covariance.

3. \[ \text{Cov}(aX + b, cY + d) = ac \text{Cov}(X, Y). \]
   \[ \rho(aX + b, cY + d) = \rho(X, Y). \]

4. Suppose that the joint distribution of $X$ and $Y$ is a uniform distribution over $x^2 + y^2 \leq 1$. Determine the correlation coefficient of $X$ and $Y$.
   **Solution.** The joint density is $f(x, y) = \frac{1}{\pi}$ for $(x, y) \in \{(x, y)|x^2 + y^2 \leq 1\}$ and 0 otherwise. The marginal density is
   \[ f_X(x) = \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}, -1 \leq x \leq 1. \]
   Since $f_X(x)$ is an even function, $x f_X(x)$ is odd so from symmetry, $\mathbb{E}X = 0$. Similarly, $\mathbb{E}Y = 0$. Also you should be able to show $\mathbb{E}(XY) = 0$. Therefore, $\rho = 0$.

5. Suppose that $X$ and $Y$ are identically distributed Bernoulli random variables with parameter $0 < p < 1$ such that $\text{Cov}(X, Y) = 0$. Show that $X$ and $Y$ are independent.
   **Solution.** The condition $\text{Cov}(X, Y) = 0$ is equivalent to $P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = p^2$. Then it follows $P(X = 1, Y = 0) = P(X = 1) - P(X = 1, Y = 1) = p(1 - p) = P(X = 1)P(Y = 0)$ and similarly for other cases. So we have $P(X = i, Y = j) = P(X = i)P(Y = j)$ for all $i, j = 0, 1$.

6. For random variables $X$ and $Y$, prove or disprove that $X$ and $Y - \mathbb{E}[Y|X]$ are correlated.
   **Solution.**
   \[
   \text{cov}(X, Y - \mathbb{E}[Y|X]) = \mathbb{E}[XY - \mathbb{E}[Y|X]] - \mathbb{E}(X)\mathbb{E}[Y - \mathbb{E}[Y|X]]
   = \mathbb{E}(XY) - \mathbb{E}[X\mathbb{E}[Y|X]] - \mathbb{E}(X)\mathbb{E}[Y] + \mathbb{E}(X)\mathbb{E}[\mathbb{E}[Y|X]]
   = \mathbb{E}(XY) - \mathbb{E}[X\mathbb{E}[Y|X]] - \mathbb{E}(X)\mathbb{E}[Y] + \mathbb{E}(X)\mathbb{E}[Y] = 0
   \]

7. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$. Then are $\bar{X}$ and $X_j - \bar{X}$ independent?
   **Solution.** $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Since $X_j - \bar{X}$ is a linear combination of $X_j$'s, it is also distributed as normal. Any uncorrelated normal distributions are independent. So we only need to check if $\text{Cov}(\bar{X}, X_j - \bar{X}) = 0$.
   \[
   \text{Cov}(\bar{X}, X_j - \bar{X}) = \text{Cov}(\frac{1}{n} \sum_{i=1}^{n} X_i, X_j - \frac{1}{n} \sum_{k=1}^{n} X_k)
   = \frac{1}{n} \sum_{i=1}^{n} \text{Cov}(X_i, X_j) - \frac{1}{n} \sum_{i,k=1}^{n} \text{Cov}(X_i, X_k)
   = \frac{1}{n} \text{Var}(X_j) - \frac{1}{n} \sum_{i=1}^{n} \text{Var}(X_i)
   = \frac{1}{n} \text{Var}(X_j) - \frac{1}{n} \frac{1}{n} \text{Var}(X_j) = 0
   \]

**Note. 1.** The second midterm exam result: mean 25.2 ± 8.6. max 40, min 5. 12 students above 30. 18 students between 20 and 30. 11 students below 20.

**Note. 2.** The final homework problem will be posted in the TA’s class website by this Thursday. Solve 9 problems in Chapter 6 and 7. Due date: December 15 (Thursday).

Figure 1: Number of students (vertical) below the given exam score (horizontal).