1. Given $n$ distinguishable objects $x_1, \ldots, x_n$, permutation is a rearrangement of the $n$ objects. There are $n!$ different rearrangements. Combination is a way to choose $k$ distinct objects out of the $n$ objects. There are $\binom{n}{k}$ different combinations. The notation is called $n$ choose $k$.

2. Binomial identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

3. Problem. A fair coin is tossed independently $n$ times ($n > 3$). Find the probability that we get 2 heads and $n-2$ tails.

4. Problem. A fair coin is tossed independently $n$ times ($n > 3$). Find the probability at least three of tosses yield heads.

Solution. Let $X$ be the number of heads.

$$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$$

$$= 1 - \frac{1}{2^n} - \binom{n}{1} \frac{1}{2^n} - \binom{n}{2} \frac{1}{2^n}.$$  

5. If $n$ identical balls are placed at random into $n$ distinct boxes, find the probability that exactly one box remains empty.

Solution. Let A be the empty box, B the box containing two balls and C’s $n-2$ boxes containing one ball. There are $2\binom{n}{2}$ ways of arranging the letter sequence $ABC \ldots C$. The size of the sample space is $\binom{n+n-1}{n}$ since it is equivalent to the number of ways of placing $n$ 0’s and $n-1$ 1’s in order. So the probability is $\frac{n(n-1)}{2^n}$.

6. If $n$ identical balls are placed into $n-1$ distinct boxes so that each distinguishable arrangement is equally likely. Find the probability that no box remains empty.

Solution. Let $A$ be the box containing two balls and $B$’s $n-2$ boxes containing one ball. There are $n-1$ ways of arranging the letter sequence $AB \cdots B$. The size of the sample space is $\binom{n+n-2}{n}$ since it is equivalent to the number of ways of placing $n$ 0’s and $n-2$ 1’s in order. We expect the equal probability for each sample point. So the probability is $\frac{n-1}{2^n}$.

Assigned Problems