1. The expected value of a discrete random variable $X$ is defined as

$$\mathbb{E}X = \sum xP(X = x) = \sum xp(x).$$

2. The expected value of function $Y = h(X)$ is

$$\mathbb{E}h(X) = \sum yP(h(X) = y) = \sum h(x)P(h(X) = h(x)) = \sum h(x)P(X = x) = \sum h(x)p(x).$$

3. $\forall X = \mathbb{E}X^2 - (\mathbb{E}X)^2. \forall (aX + b) = a^2\mathbb{V}X.$

4. For $X \sim Bernoulli(p), \mathbb{E}X = p$ and $\mathbb{V}X = p.$

5. Consider an experiment that give only two outcomes $H$ or $T$. Let $P(H) = p$ and $P(T) = 1 - p$. This is a Bernoulli experiment. Now consider $n$ independent experiments where each experiment gives only two outcomes. This is a Binomial experiment. Let $X$ be the number of outcome $H$. This is called the Binomial random variable with parameter $n$ and $p$. It will be denoted by $X \sim Binomial(n, p).$ Note that if $X_i$ is the number of outcome $H$ in the $i$-th experiment, $X = X_1 + X_2 + \cdots + X_n$.

6. Problem. Find a value $c$ that minimizes $\mathbb{E}(X - c)^2$ for a discrete random variable $X$.

solution. Let $f(c) = \mathbb{E}(X - c)^2 = c^2 - (2\mathbb{E}X)c + \mathbb{E}X^2$. Then $f(c)$ is minimum at a point where $f'(c) = 0$. $f'(c) = 2c - 2\mathbb{E}X = 0$. So $c = \mathbb{E}X$ will give the minimum.

7. Problem. A man with $n$ keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean and the variance of the number of trials if unsuccessful keys are eliminated.

solution. Let $X$ be the number of trials. Then $P(X = 1) = \frac{1}{n}, P(X = 2) = \frac{n-1}{n} \frac{1}{n-1} = \frac{1}{n}, \cdots, P(X = n) = \frac{n-1}{n} \cdots \frac{1}{2} = \frac{1}{n}$. The random variable $X$ is said to be uniformly distributed (or a discrete uniform random variable).

$$\mathbb{E}X = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

$$\mathbb{E}X^2 = \sum_{x=1}^{n} \frac{x^2}{n} = \frac{1}{n} \frac{n(n+1)(2n+1)}{6}.$$

So $\text{Var}X = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$.

8. Problem. For $X \sim Binomial(n, p)$, compute $\mathbb{E}(X^3)$.

Solution. From p. 104 of your textbook, $\mathbb{E}X(X - 1) = n(n - 1)p^2$. Note that $\mathbb{E}X^3 = \mathbb{E}(X - 1)(X - 2) - 3\mathbb{E}X^2 + 2\mathbb{E} = n(n - 1)(n - 2)p^3$. 

**Assigned Problems** Exercise 3.38., Exercise 3.50. Second Homework due on Feb 10. 9:30AM. Submit solutions to assigned problems from Lecture 4-7 (8 problems).