1. A box contains 7 white balls and 3 black balls. A ball is chosen at random and replaced by the opposite color. Now again draw a ball.

(a) What is the probability that the second ball drawn is white?

Solution. Let $W_i$ be the event that the $i$-th ball is white and $B_i$ be the event that the $i$-th ball is black.

$$P(W_2) = P(W_2|W_1)P(W_1) + P(W_2|B_1)P(B_1) = \frac{6}{10} \cdot \frac{7}{10} + \frac{8}{10} \cdot \frac{3}{10} = 0.66$$

(b) What is the probability that the first ball drawn is black when the second ball drawn is white?

Solution. $P(B_1|W_2) = \frac{P(W_2|B_1)P(B_1)}{P(W_2)} = \frac{\frac{8}{10} \cdot \frac{3}{10}}{\frac{6}{10} + \frac{8}{10}} = \frac{12}{33}$.

2. The joint distribution of $X$ and $Y$ is given by $f(x, y) = 2$ for $y < x < 1$, $0 < y < 1$ otherwise.

(a) Compute the probability $P\left(\frac{1}{2} < X < 1, \frac{1}{2} < Y < 1\right)$.

Solution. Since it is a uniform distribution, the probability is equivalent to the area bounded by $\frac{1}{2} < X < 1$ and $\frac{1}{2} < Y < 1$ times 2 so it is $\frac{1}{4}$.

(b) Determine the marginal distributions of $X$ and $Y$.

Solution. $f_X(x) = \int_x^1 2 \, dy = 2 - 2x$ for $0 < x < 1$. $f_Y(x) = \int_0^y 2 \, dx = 2y$ for $0 < y < 1$.

(c) Are $X$ and $Y$ are independent? Prove your statement.

Solution. $f\left(\frac{3}{4}, \frac{3}{4}\right) \neq f_X\left(\frac{3}{4}\right)f_Y\left(\frac{3}{4}\right)$. Hence they are not independent.

3. A random sample $X_1, X_2, \cdots, X_n$ comes from the distribution $f(x) = \lambda x^{\lambda - 1}, 0 < x < 1$.

(a) Write down the likelihood function for $\lambda$ clearly.

Solution. $L(\lambda) = \lambda^n (X_1 \cdot X_2 \cdots X_n)^{\lambda - 1}$.

(b) Derive the maximum likelihood estimator of $\theta$.

Solution. The log-likelihood is $\ln L(\lambda) = n \ln \lambda + (\lambda - 1) \sum_{i=1}^n \ln X_i$. Differentiating with respect to $\lambda$, we get $\frac{\partial \ln L(\lambda)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \ln X_i = 0$. Hence $\hat{\lambda} = -\frac{n}{\sum_{i=1}^n \ln X_i}$.
4. The intelligence quotient (IQ) of 10 dogs are measured:

\[25, 21, 22, 17, 29, 25, 16, 20, 19, 22.\]

The measurements are assumed to follow a normal distribution with known variance 9.

The following R output is available.

\[> \text{qnorm(c(0.025, 0.05, 0.1))}\]

\[[1] -1.96 \quad -1.64 \quad -1.28\]

(a) Construct a 95\% confidence interval of the mean IQ.

**Solution.** Since \(\sigma = 3\) is known, a CI is based on the Z-statistic: 
\[z = (\bar{x} - \mu) / (\sigma / \sqrt{n}).\]
CI is given by \(\bar{x} \pm z_{\alpha/2}3/\sqrt{10}\). The final answer is \(19.7 \leq \mu \leq 23.5\).

(b) Determine the minimum sample size needed for the width of a 95\% confidence interval to be less than 2.

**Solution.** We need 
\[2z_{\alpha/2}3/\sqrt{n} \leq 2.\]
Solving this, we need \((1.96 \cdot 3)^2 = 34.57 \leq n\). Hence 
\(n = 35\).