Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathcal{P}}$ and let $\bar{\mathcal{P}}_U$ be the family of distributions of $U$ as $P$ ranges over $\bar{\mathcal{P}}$. A test is said to have Neyman structure w.r.t. $U$ if

$$E[T(X) | U] = \alpha \quad \text{a.s. } \bar{\mathcal{P}}_U,$$

Clearly, if $T$ has Neyman structure, then

$$E[T(X)] = E\{E[T(X) | U]\} = \alpha \quad P \in \bar{\mathcal{P}},$$

i.e., $T$ is similar on $\bar{\Theta}_{01}$. If all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. $U$, then working with tests having Neyman structure is the same as working with tests similar on $\bar{\Theta}_{01}$.

**Lemma 6.6**

Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathcal{P}}$. A necessary and sufficient condition for all tests similar on $\bar{\Theta}_{01}$ to have Neyman structure w.r.t. $U$ is that $U$ is boundedly complete for $P \in \bar{\mathcal{P}}$. 
Proof

(i) Suppose first that \( U \) is boundedly complete for \( P \in \bar{\mathcal{P}} \).
Let \( T(X) \) be a test similar on \( \bar{\Theta}_{01} \).
Then \( E[T(X) - \alpha] = 0 \) for all \( P \in \bar{\mathcal{P}} \).
From the boundedness of \( T(X) \), \( E[T(X)|U] \) is bounded.
Since \( E\{E[T(X)|U] - \alpha\} = E[T(X) - \alpha] = 0 \) for all \( P \in \bar{\mathcal{P}} \) and \( U \) is
boundedly complete, \( E[T(X)|U] = \alpha \) a.s. \( \bar{\mathcal{P}}_U \), i.e., \( T \) has Neyman
structure.

(ii) Suppose now that all tests similar on \( \bar{\Theta}_{01} \) have Neyman structure
w.r.t. \( U \).
Suppose also that \( U \) is not boundedly complete for \( P \in \bar{\mathcal{P}} \).
Then there is a function \( h \) such that \( |h(u)| \leq C \), \( E[h(U)] = 0 \) for all \( P \in \bar{\mathcal{P}} \), and \( h(U) \neq 0 \) with positive probability for some \( P \in \bar{\mathcal{P}} \).
Let \( T(X) = \alpha + ch(U) \), where \( c = \min\{\alpha, 1 - \alpha\}/C \).
Then \( T \) is a test similar on \( \bar{\Theta}_{01} \) but \( T \) does not have Neyman structure
w.r.t. \( U \) (because \( h(U) \neq 0 \)).
Thus, \( U \) must be boundedly complete for \( P \in \bar{\mathcal{P}} \).
This proves the result.
**Theorem 6.4 (UMPU tests in multiparameter exponential families)**

Suppose that $X$ has the following p.d.f. w.r.t. a $\sigma$-finite measure:

$$f_{\theta,\varphi}(x) = \exp \{ \theta Y(x) + \varphi^\tau U(x) - \zeta(\theta, \varphi) \},$$

where $\theta$ is a real-valued parameter, $\varphi$ is a vector-valued parameter, and $Y$ (real-valued) and $U$ (vector-valued) are statistics.

(i) For testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, a UMPU test of size $\alpha$ is

$$T^*_*(Y, U) = \begin{cases} 1 & Y > c(U) \\ \gamma(U) & Y = c(U) \\ 0 & Y < c(U), \end{cases}$$

where $c(u)$ and $\gamma(u)$ are Borel functions determined by

$$E_{\theta_0}[T^*_*(Y, U)|U = u] = \alpha \text{ for every } u$$

and $E_{\theta_0}$ is the expectation w.r.t. $f_{\theta_0,\varphi}$.

(ii) For testing $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ versus $H_1 : \theta_1 < \theta < \theta_2$, a UMPU test of size $\alpha$ is

$$T^*_*(Y, U) = \begin{cases} 1 & c_1(U) < Y < c_2(U) \\ \gamma_i(U) & Y = c_i(U), \ i = 1, 2, \\ 0 & Y < c_1(U) \text{ or } Y > c_2(U), \end{cases}$$
Theorem 6.4 (continued)

where $c_i(u)$’s and $\gamma_i(u)$’s are Borel functions determined by

$$E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha$$ for every $u$.

(iii) For testing $H_0: \theta_1 \leq \theta \leq \theta_2$ versus $H_1: \theta < \theta_1$ or $\theta > \theta_2$, a UMPU test of size $\alpha$ is

$$T_*(Y, U) = \begin{cases} 
1 & Y < c_1(U) \text{ or } Y > c_2(U) \\
\gamma_i(U) & Y = c_i(U), i = 1, 2, \\
0 & c_1(U) < Y < c_2(U),
\end{cases}$$

where $c_i(u)$’s and $\gamma_i(u)$’s are Borel functions determined by

$$E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha$$ for every $u$.

(iv) For testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, a UMPU test of size $\alpha$ is given by $T_*(Y, U)$ in (iii), where $c_i(u)$’s and $\gamma_i(u)$’s are Borel functions determined by

$$E_{\theta_0}[T_*(Y, U)|U = u] = \alpha$$ for every $u$

and

$$E_{\theta_0}[T_*(Y, U) Y|U = u] = \alpha E_{\theta_0}(Y|U = u)$$ for every $u$.  

By sufficiency, we only need to consider tests that are functions of \((Y, U)\).

It follows from Theorem 2.1(i) that the p.d.f. of \((Y, U)\) (w.r.t. a \(\sigma\)-finite measure) is in a natural exponential family of the form
\[
\exp\{\theta y + \varphi^\top u - \zeta(\theta, \varphi)\}
\]
and, given \(U = u\), the p.d.f. of the conditional distribution of \(Y\) (w.r.t. a \(\sigma\)-finite measure \(\nu_u\)) is in a natural exponential family of the form
\[
\exp\{\theta y - \zeta_u(\theta)\}.
\]

Hypotheses in (i)-(iv) are of the form \(H_0 : \theta \in \Theta_0 \text{ vs } H_1 : \theta \in \Theta_1\) with
\[
\bar{\Theta}_{01} = \{(\theta, \varphi) : \theta = \theta_0\} \text{ or } = \{(\theta, \varphi) : \theta = \theta_i, i = 1, 2\}.
\]

In case (i) or (iv), \(U\) is sufficient and complete for \(P \in \bar{\mathcal{P}}\) and, hence, Lemma 6.6 applies.

In case (ii) or (iii), applying Lemma 6.6 to each \(\{(\theta, \varphi) : \theta = \theta_i\}\) also shows that working with tests having Neyman structure is the same as working with tests similar on \(\bar{\Theta}_{01}\).

By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.
Thus, for (i), it suffices to show $T_\ast$ is UMP among all tests $T$ satisfying
\begin{equation}
E_{\theta_0}[T(Y, U)|U = u] = \alpha \text{ for every } u
\end{equation}

and for part (ii) or (iii), it suffices show $T_\ast$ is UMP among all tests $T$ satisfying
\begin{equation}
E_{\theta_1}[T(Y, U)|U = u] = E_{\theta_2}[T(Y, U)|U = u] = \alpha \text{ for every } u.
\end{equation}

For (iv), any unbiased $T$ should satisfy (1) and
\begin{equation}
\frac{\partial}{\partial \theta} E_{\theta, \varphi}[T(Y, U)] = 0, \quad \theta \in \bar{\Theta}_{01}. \quad (2)
\end{equation}

One can show (exercise) that (2) is equivalent to
\begin{equation}
E_{\theta, \varphi}[T(Y, U)Y - \alpha Y] = 0, \quad \theta \in \bar{\Theta}_{01}. \quad (3)
\end{equation}

Using the argument in the proof of Lemma 6.6, one can show (exercise) that (3) is equivalent to
\begin{equation}
E_{\theta_0}[T(Y, U)Y|U = u] = \alpha E_{\theta_0}(Y|U = u) \text{ for every } u. \quad (4)
\end{equation}

Hence, for (iv), it suffices to show $T_\ast$ is UMP among all tests $T$ satisfying (1) and (4).
Note that the power function of any test $T(Y, U)$ is
\[ \beta_T(\theta, \varphi) = \int \left[ \int T(y, u) dP_{Y|U=u}(y) \right] dP_U(u). \]
Thus, it suffices to show that for every fixed $u$ and $\theta \in \Theta_1$, $T_*$ maximizes
\[ \int T(y, u) dP_{Y|U=u}(y) \]
over all $T$ subject to the given side conditions.
Since $P_{Y|U=u}$ is in a one-parameter exponential family, the results in (i) and (ii) follow from Corollary 6.1 and Theorem 6.3, respectively.
The result in (iii) follows from Theorem 6.3(ii) by considering $1 - T_*$. To prove the result in (iv), it suffices to show that if $Y$ has the p.d.f. given by $\exp \{ \theta y - \zeta_u(\theta) \}$ and if $u$ is treated as a constant in (1) and (4), $T_*$ in (iii) with a fixed $u$ is UMP subject to conditions (1) and (4).
We now omit $u$ in the following proof for (iv), which is very similar to the proof of Theorem 6.3.
First, $(\alpha, \alpha E_{\theta_0}(Y))$ is an interior point of the set of points $(E_{\theta_0}[T(Y)], E_{\theta_0}[T(Y)Y])$ as $T$ ranges over all tests of the form $T(Y)$. 
By Lemma 6.2 and Proposition 6.1, for testing $\theta = \theta_0$ versus $\theta = \theta_1$, the UMP test is equal to 1 when

$$(k_1 + k_2 y) e^{\theta_0 y} < C(\theta_0, \theta_1) e^{\theta_1 y},$$

where $k_i$'s and $C(\theta_0, \theta_1)$ are constants. This inequality is equivalent to

$$a_1 + a_2 y < e^{by}$$

for some constants $a_1$, $a_2$, and $b$.

This region is either one-sided or the outside of an interval.

By Theorem 6.2(ii), a one-sided test has a strictly monotone power function and therefore cannot satisfy (4).

Thus, this test must have the form of $T_*$ in (iii).

Since $T_*$ in (iii) does not depend on $\theta_1$, by Lemma 6.1, it is UMP over all tests satisfying (1) and (4); in particular, the test $\equiv \alpha$.

Thus, $T_*$ is UMPU.

Finally, it can be shown that all the $c$- and $\gamma$-functions in (i)-(iv) are Borel functions of $u$ (see Lehmann (1986, p. 149)).
Example 6.11

A problem arising in many different contexts is the comparison of two treatments.

If the observations are integer-valued, the problem often reduces to testing the equality of two Poisson distributions (e.g., a comparison of the radioactivity of two substances or the car accident rate in two cities) or two binomial distributions (when the observation is the number of successes in a sequence of trials for each treatment).

Consider first the Poisson problem in which $X_1$ and $X_2$ are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$.

The p.d.f. of $X = (X_1, X_2)$ is

$$[e^{-(\lambda_1+\lambda_2)} / x_1! x_2!] \exp \{ x_2 \log(\lambda_2 / \lambda_1) + (x_1 + x_2) \log \lambda_1 \}$$

w.r.t. the counting measure on \{(i,j) : i = 0,1,2,\ldots, j = 0,1,2,\ldots\}.

The hypotheses such as $\lambda_1 = \lambda_2$ and $\lambda_1 \geq \lambda_2$ are equivalent to $\theta = 0$ and $\theta \leq 0$, respectively, where $\theta = \log(\lambda_2 / \lambda_1)$.

The p.d.f. of $X$ is in a multiparameter exponential family with $\varphi = \log \lambda_1$, $Y = X_2$, and $U = X_1 + X_2$. 
Thus, Theorem 6.4 applies.

To obtain various tests in Theorem 6.4, it is enough to derive the conditional distribution of \( Y = X_2 \) given \( U = X_1 + X_2 = u \).

Using the fact that \( X_1 + X_2 \) has the Poisson distribution \( P(\lambda_1 + \lambda_2) \), one can show that

\[
P(Y = y | U = u) = \binom{u}{y} p^y (1 - p)^{u-y} 1_{\{0,1,...,u\}}(y), \quad u = 0, 1, 2, ..., \]

where \( p = \lambda_2 / (\lambda_1 + \lambda_2) = e^\theta / (1 + e^\theta) \).

This is the binomial distribution \( Bi(p, u) \).

On the boundary set \( \tilde{\Theta}_{01} \), \( \theta = \theta_j \) (a known value) and the distribution \( P_{Y|U=u} \) is known.

Consider next the binomial problem in which \( X_j, j = 1, 2, \) are independently distributed as the binomial distributions \( Bi(p_j, n_j) \), \( j = 1, 2, \) respectively, where \( n_j \)'s are known but \( p_j \)'s are unknown.

The p.d.f. of \( X = (X_1, X_2) \) is

\[
\binom{n_1}{x_1} \binom{n_2}{x_2} (1 - p_1)^{n_1} (1 - p_2)^{n_2} \exp \left\{ x_2 \log \frac{p_2(1-p_1)}{p_1(1-p_2)} + (x_1 + x_2) \log \frac{p_1}{1-p_1} \right\}
\]
w.r.t. the counting measure on \( \{(i,j) : i = 0, 1, \ldots, n_1, j = 0, 1, \ldots, n_2\} \).

This p.d.f. is in a multiparameter exponential family with \( \theta = \log \frac{p_2(1-p_1)}{p_1(1-p_2)}, \ Y = X_2 \), and \( U = X_1 + X_2 \).

Thus, Theorem 6.4 applies.

Note that hypotheses such as \( p_1 = p_2 \) and \( p_1 \geq p_2 \) are equivalent to \( \theta = 0 \) and \( \theta \leq 0 \), respectively.

Using the joint distribution of \((X_1, X_2)\), one can show (exercise) that

\[
P(Y = y | U = u) = K_u(\theta) \left( \binom{n_1}{u-y} \binom{n_2}{y} e^{\theta y} I_A(y) \right), \quad u = 0, 1, \ldots, n_1 + n_2,
\]

where

\[
A = \{y : y = 0, 1, \ldots, \min\{u, n_2\}, u - y \leq n_1\}
\]

and

\[
K_u(\theta) = \left[ \sum_{y \in A} \binom{n_1}{u-y} \binom{n_2}{y} e^{\theta y} \right]^{-1}.
\]

If \( \theta = 0 \), this distribution reduces to a known distribution: the hypergeometric distribution \( HG(u, n_2, n_1) \) (Table 1.1, page 18).
The following lemma is useful especially when \( X \) is from a population in an exponential family with continuous p.d.f.'s.

**Lemma 6.7**

Suppose that \( X \) has the following p.d.f. w.r.t. a \( \sigma \)-finite measure:

\[
f_{\theta, \varphi}(x) = \exp \left\{ \theta Y(x) + \varphi^\tau U(x) - \zeta(\theta, \varphi) \right\},
\]

where \( \theta \) is a real-valued parameter, \( \varphi \) is a vector-valued parameter, and \( Y \) (real-valued) and \( U \) (vector-valued) are statistics. Let \( V(Y, U) \) be a statistic independent of \( U \) when \( \theta = \theta_j \), where \( \theta_j \)'s are known values given in the hypotheses in (i)-(iv) of Theorem 6.4.

(i) If \( V(y, u) \) is increasing in \( y \) for each \( u \), then the UMPU tests in (i)-(iii) of Theorem 6.4 are equivalent to those with \( Y \) and \( (Y, U) \) replaced by \( V \) and with \( c_i(U) \) and \( \gamma_i(U) \) replaced by constants \( c_i \) and \( \gamma_i \), respectively.

(ii) If there are Borel functions \( a(u) > 0 \) and \( b(u) \) such that \( V(y, u) = a(u)y + b(u) \), then the UMPU test in Theorem 6.4(iv) is equivalent to that with \( Y \) and \( (Y, U) \) replaced by \( V \) and with \( c_i(U) \) and \( \gamma_i(U) \) replaced by constants \( c_i \) and \( \gamma_i \), respectively.
Proof

(i) Since $V$ is increasing in $y$, $Y > c_i(u)$ is equivalent to $V > d_i(u)$ for some $d_i$.

The result follows from the fact that $V$ is independent of $U$ so that $d_i$’s and $\gamma_i$’s do not depend on $u$ when $Y$ is replaced by $V$.

(ii) Since $V = a(U)Y + b(U)$, the UMPU test in Theorem 6.4(iv) is the same as

\[
T^*_1(V, U) = \begin{cases} 
1 & V < c_1(U) \text{ or } V > c_2(U) \\
\gamma_i(U) & V = c_i(U), \ i = 1, 2, \\
0 & c_1(U) < V < c_2(U),
\end{cases}
\]

subject to $E_{\theta_0}[T^*_1(V, U)|U = u] = \alpha$ and

\[
E_{\theta_0} \left[ T^*_1(V, U) \frac{V - b(U)}{a(U)} | U \right] = \alpha E_{\theta_0} \left[ \frac{V - b(U)}{a(U)} | U \right]. 
\] (5)

Under $E_{\theta_0}[T^*_1(V, U)|U = u] = \alpha$, (5) is the same as

$E_{\theta_0}[T^*_1(V, U)V|U] = \alpha E_{\theta_0}(V|U)$.

Since $V$ and $U$ are independent when $\theta = \theta_0$, $c_i(u)$’s and $\gamma_i(u)$’s do not depend on $u$ and, therefore, $T^*_1$ does not depend on $U$. 
If the conditions of Lemma 6.7 are satisfied, then UMPU tests can be derived by working with the distribution of $V$ instead of $P_{Y|U=u}$.

In exponential families, a $V(Y, U)$ independent of $U$ can often be found by applying Basu’s theorem (Theorem 2.4).

An important application of Theorem 6.4 and Lemma 6.7 is the derivation of UMPU tests in normal families.

The results presented here are the basic justifications for tests in elementary textbooks concerning parameters in normal families.

When we consider normal families, $\gamma_i$'s can be chosen to be 0 since the c.d.f. of $Y$ given $U = u$ or the c.d.f. of $V$ is continuous.

### One-sample problems

Let $X_1, ..., X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with unknown $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, where $n \geq 2$.

The joint p.d.f. of $X = (X_1, ..., X_n)$ is

$$
\frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 + \frac{\mu}{\sigma^2} \sum_{i=1}^{n} x_i - \frac{n\mu^2}{2\sigma^2} \right\}.
$$
Tests concerning $\sigma^2$

Consider first hypotheses concerning $\sigma^2$.
The p.d.f. of $X$ is in a multiparameter exponential family with
\[ \theta = -(2\sigma^2)^{-1}, \quad \varphi = n\mu / \sigma^2, \quad Y = \sum_{i=1}^{n} X_i^2, \quad \text{and} \quad U = \bar{X}. \]
By Basu’s theorem, $V = (n-1)S^2$ is independent of $U = \bar{X}$ (Example 2.18), where $S^2$ is the sample variance.
Also,
\[ \sum_{i=1}^{n} X_i^2 = (n-1)S^2 + n\bar{X}^2, \]
i.e., $V = Y - nU^2$.
Hence the conditions of Lemma 6.7 are satisfied.
Since $V/\sigma^2$ has the chi-square distribution $\chi^2_{n-1}$ (Example 2.18),
values of $c_i$’s for hypotheses in (i)-(iii) of Theorem 6.4 are related to quantiles of $\chi^2_{n-1}$.
For testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ (which is equivalent to testing
$H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$), $d_i = c_i / \sigma_0^2, \ i = 1, 2,$ are determined by
\[ \int_{d_1}^{d_2} f_{n-1}(v)dv = 1 - \alpha \quad \text{and} \quad \int_{d_1}^{d_2} vf_{n-1}(v)dv = (n-1)(1 - \alpha), \]
where \( f_m \) is the Lebesgue p.d.f. of the chi-square distribution \( \chi^2_m \). Since \( \nu f_{n-1}(\nu) = (n-1)f_{n+1}(\nu) \), \( d_1 \) and \( d_2 \) are determined by

\[
\int_{d_1}^{d_2} f_{n-1}(\nu) d\nu = \int_{d_1}^{d_2} f_{n+1}(\nu) d\nu = 1 - \alpha.
\]

If \( n - 1 \approx n + 1 \), then \( d_1 \) and \( d_2 \) are nearly the \((\alpha/2)\)th and \((1 - \alpha/2)\)th quantiles of \( \chi^2_{n-1} \), respectively, in which case the UMPU test in Theorem 6.4(iv) is the same as the “equal-tailed” chi-square test for \( H_0 \) in elementary textbooks.

Tests concerning \( \mu \)

Consider next hypotheses concerning \( \mu \).

The p.d.f. of \( X \) has is in a multiparameter exponential family with

\[ Y = \bar{X}, \quad U = \sum_{i=1}^{n} (X_i - \mu_0)^2, \quad \theta = n(\mu - \mu_0)/\sigma^2, \quad \phi = -(2\sigma^2)^{-1}. \]

For testing hypotheses \( H_0 : \mu \leq \mu_0 \) versus \( H_1 : \mu > \mu_0 \), we take \( V \) to be

\[ t(X) = \sqrt{n}(\bar{X} - \mu_0)/S. \]

By Basu’s theorem, \( t(X) \) is independent of \( U \) when \( \mu = \mu_0 \). Hence it satisfies the conditions in Lemma 6.7(i).
From Examples 1.16 and 2.18, \( t(X) \) has the t-distribution \( t_{n-1} \) when \( \mu = \mu_0 \). Thus, \( c(U) \) in Theorem 6.4(i) is the \((1 - \alpha)\)th quantile of \( t_{n-1} \).

For the two-sided hypotheses \( H_0 : \mu = \mu_0 \) versus \( H_1 : \mu \neq \mu_0 \), the statistic \( V = (\bar{X} - \mu_0)/\sqrt{U} \) satisfies the conditions in Lemma 6.7(ii) and has a distribution symmetric about 0 when \( \mu = \mu_0 \). Then the UMPU test in Theorem 6.4(iv) rejects \( H_0 \) when \( |V| > d \), where \( d \) satisfies \( P(|V| > d) = \alpha \) when \( \mu = \mu_0 \).

Since
\[
t(X) = \sqrt{(n-1)nV(X)/\sqrt{1 - n[V(X)]^2}},
\]
the UMPU test rejects \( H_0 \) if and only if \( |t(X)| > t_{n-1,\alpha/2} \), where \( t_{n-1,\alpha} \) is the \((1 - \alpha)\)th quantile of the t-distribution \( t_{n-1} \).

The UMPU tests derived here are the so-called one-sample t-tests in elementary textbooks. The power function of a one-sample t-test is related to the noncentral t-distribution introduced in §1.3.1 (see Exercise 36).