Another popular method of constructing confidence sets is to use a close relationship between confidence sets and hypothesis tests. For any test $T$, the set $\{x : T(x) \neq 1\}$ is called the acceptance region. This terminology is not precise when $T$ is a randomized test.

**Theorem 7.2**

For each $\theta_0 \in \Theta$, let $T_{\theta_0}$ be a test for $H_0 : \theta = \theta_0$ (versus some $H_1$) with significance level $\alpha$ and acceptance region $A(\theta_0)$. For each $x$ in the range of $X$, define

$$C(x) = \{\theta : x \in A(\theta)\}.$$  

Then $C(X)$ is a level $1 - \alpha$ confidence set for $\theta$. If $T_{\theta_0}$ is nonrandomized and has size $\alpha$ for every $\theta_0$, then $C(X)$ has confidence coefficient $1 - \alpha$. 

Proof

We prove the first assertion only. The proof for the second assertion is similar. Under the given condition,

\[
\sup_{\theta=\theta_0} P(X \not\in A(\theta_0)) = \sup_{\theta=\theta_0} P(T_{\theta_0} = 1) \leq \alpha,
\]

which is the same as

\[
1 - \alpha \leq \inf_{\theta=\theta_0} P(X \in A(\theta_0)) = \inf_{\theta=\theta_0} P(\theta_0 \in C(X)).
\]

Since this holds for all \( \theta_0 \), the result follows from

\[
\inf_{P \in \mathcal{P}} P(\theta \in C(X)) = \inf_{\theta_0 \in \Theta} \inf_{\theta=\theta_0} P(\theta_0 \in C(X)) \geq 1 - \alpha.
\]

The converse of Theorem 7.2 is partially true.
Proposition 7.2

Let \( C(X) \) be a confidence set for \( \theta \) with confidence level (or confidence coefficient) \( 1 - \alpha \).

For any \( \theta_0 \in \Theta \), define a region \( A(\theta_0) = \{ x : \theta_0 \in C(x) \} \).

Then the test \( T(X) = 1 - I_{A(\theta_0)}(X) \) has significance level \( \alpha \) for testing \( H_0 : \theta = \theta_0 \) versus some \( H_1 \).

Discussions

In general, \( C(X) \) in Theorem 7.2 can be determined numerically, if it does not have an explicit form.

Suppose \( A(\theta) = \{ Y : a(\theta) \leq Y \leq b(\theta) \} \) for a real-valued \( \theta \) and statistic \( Y(X) \) and some nondecreasing functions \( a(\theta) \) and \( b(\theta) \).

When we observe \( Y = y \), \( C(X) \) is an interval with limits \( \underline{\theta} \) and \( \overline{\theta} \), which are the \( \theta \)-values at which the horizontal line \( Y = y \) intersects the curves \( Y = b(\theta) \) and \( Y = a(\theta) \) (Figure 7.1), respectively.

If \( y = b(\theta) \) (or \( y = a(\theta) \)) has no solution or more than one solution, \( \underline{\theta} = \inf\{ \theta : y \leq b(\theta) \} \) (or \( \overline{\theta} = \sup\{ \theta : a(\theta) \leq y \} \)).

\( C(X) \) does not include \( \underline{\theta} \) (or \( \overline{\theta} \)) if and only if at \( \underline{\theta} \) (or \( \overline{\theta} \)), \( b(\theta) \) (or \( a(\theta) \)) is only left-continuous (or right-continuous).
Figure 7.1. A confidence interval obtained by inverting $A(\theta) = [a(\theta), b(\theta)]$.
Example 7.7

Suppose that $X$ has the following p.d.f. in a one-parameter exponential family:

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x),$$

where $\theta$ is real-valued and $\eta(\theta)$ is nondecreasing in $\theta$.

First, we apply Theorem 7.2 with $H_0 : \theta = \theta_0$ and $H_1 : \theta > \theta_0$.

By Theorem 6.2, the acceptance region of the UMP test of size $\alpha$ is

$$A(\theta_0) = \{x : Y(x) \leq c(\theta_0)\},$$

where $c(\theta_0) = c$ in Theorem 6.2.

It can be shown that $c(\theta)$ is nondecreasing in $\theta$.

Inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c(\theta)$ and $a(\theta)$ ignored, we obtain

$$C(X) = [\underline{\theta}(X), \infty) \quad \text{or} \quad (\bar{\theta}(X), \infty),$$

a one-sided confidence interval for $\theta$ with confidence level $1 - \alpha$.

$\underline{\theta}(X)$ is a called a lower confidence bound for $\theta$ in §2.4.3.

When the c.d.f. of $Y(X)$ is continuous, $C(X)$ has confidence coefficient $1 - \alpha$. 
If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{ \theta : Y(X) \geq c(\theta) \}$ and is of the form

$$(-\infty, \bar{\theta}(X)] \text{ or } (-\infty, \bar{\theta}(X)).$$

$\bar{\theta}(X)$ is called an upper confidence bound for $\theta$.

Consider next $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$.

By Theorem 6.4, the acceptance region of the UMPU test of size $\alpha$ is given by $A(\theta_0) = \{ x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0) \}$, where $c_i(\theta)$ are nondecreasing (exercise).

A confidence interval can be obtained by inverting $A(\theta)$ according to Figure 7.1 with $a(\theta) = c_1(\theta)$ and $b(\theta) = c_2(\theta)$.

Let us consider a specific example in which $X_1, \ldots, X_n$ are i.i.d. binary random variables with $p = P(X_i = 1)$.

Note that $Y(X) = \sum_{i=1}^n X_i$.

Suppose that we need a lower confidence bound for $p$ so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$. 
From Example 6.2, the acceptance region of a UMP test of size \( \alpha \in (0, 1) \) is \( A(p_0) = \{ y : y \leq m(p_0) \} \), where \( m(p_0) \) is an integer between 0 and \( n \) such that

\[
\sum_{j=m(p_0)+1}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j} \leq \alpha < \sum_{j=m(p_0)}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j}.
\]

Thus, \( m(p) \) is an integer-valued, left-continuous, nondecreasing step-function of \( p \).

Define

\[
\underline{p} = \inf \{ p : m(p) \geq y \} = \inf \left\{ p : \sum_{j=y}^{n} \binom{n}{j} p^j (1-p)^{n-j} > \alpha \right\}.
\]

We want to show that a level \( 1 - \alpha \) confidence interval for \( p \) is \((\underline{p}, 1]\). Inverting \( A(p) \) we obtain that

\[
C(y) = \{ p : y \leq m(p) \}.
\]
We need to show that

\[ \{ p : y \leq m(p) \} = \{ p : p < p \} \]

Suppose that \( p < p \).
If \( m(p) < y \), then, by the definition of \( p \), we must have \( p \leq p \), a contradiction.
Hence, we must have \( y \leq m(p) \).
This shows

\[ \{ p : p < p \} \subset \{ p : y \leq m(p) \} \]

Suppose that \( y \leq m(p) \).
By the definition of \( p \), \( p \leq p \).
But we cannot have \( p = p \), because \( m(p) \) is left-continuous and flat, i.e., if \( y \leq m(p) \), then there is a \( p < p \) such that \( y \leq m(p) \).
Thus, \( p < p \) and, hence,

\[ \{ p : y \leq m(p) \} \subset \{ p : p < p \} \]

One can compare this confidence interval with the one obtained by applying Theorem 7.1 (exercise).
See also Example 7.16.
Example 7.8

Suppose that $X$ has the following p.d.f. in a multiparameter exponential family:

$$f_{\theta, \varphi}(x) = \exp \{ \theta Y(x) + \varphi^\tau U(x) - \zeta(\theta, \varphi) \}$$

By Theorem 6.4, the acceptance region of a UMPU test of size $\alpha$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ or $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is

$$A(\theta_0) = \{(y, u) : y \leq c_2(u, \theta_0)\}$$

or

$$A(\theta_0) = \{(y, u) : c_1(u, \theta_0) \leq y \leq c_2(u, \theta_0)\},$$

where $c_i(u, \theta)$, $i = 1, 2$, are nondecreasing functions of $\theta$. Confidence intervals for $\theta$ can then be obtained by inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c_2(u, \theta)$ and $a(\theta) = c_1(u, \theta)$ or $a(\theta) \equiv -\infty$, for any observed $u$.

Consider more specifically the case where $X_1$ and $X_2$ are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively, and we need a lower confidence bound for the ratio $\rho = \lambda_2/\lambda_1$. 
From Example 6.11, a UMPU test of size \( \alpha \) for testing \( H_0 : \rho = \rho_0 \) versus \( H_1 : \rho > \rho_0 \) has the acceptance region

\[
A(\rho_0) = \{ (y, u) : y \leq c(u, \rho_0) \},
\]

where \( c(u, \rho_0) \) is determined by the conditional distribution of \( Y = X_2 \) given \( U = X_1 + X_2 = u \).

Since the conditional distribution of \( Y \) given \( U = u \) is the binomial distribution \( Bi(\rho/(1 + \rho), u) \), we can use the result in Example 7.7, i.e., \( c(u, \rho) \) is the same as \( m(p) \) in Example 7.7 with \( n = u \) and \( p = \rho/(1 + \rho) \).

Then a level \( 1 - \alpha \) lower confidence bound for \( p \) is \( \underline{p} \) given by

\[
\underline{p} = \inf\{ p : m(p) \geq y \} = \inf \left\{ p : \sum_{j=y}^{u} \binom{u}{j} p^j (1-p)^{u-j} \geq \alpha \right\}
\]

Since \( \rho = p/(1 - p) \) is a strictly increasing function of \( p \), a level \( 1 - \alpha \) lower confidence bound for \( \rho \) is \( \underline{p}/(1 - \underline{p}) \).
Confidence sets related to optimal tests

For a confidence set obtained by inverting the acceptance regions of some UMP or UMPU tests, it is expected that the confidence set inherits some optimality property.

**Definition 7.2**

Let \( \theta \in \Theta \) be an unknown parameter and \( \Theta' \) be a subset of \( \Theta \) that does not contain the true parameter value \( \theta \).

A confidence set \( C(X) \) for \( \theta \) with confidence coefficient \( 1 - \alpha \) is said to be \( \Theta' \)-uniformly most accurate (UMA) iff for any other confidence set \( C_1(X) \) with confidence level \( 1 - \alpha \),

\[
P(\theta' \in C(X)) \leq P(\theta' \in C_1(X)) \quad \text{for all } \theta' \in \Theta'.
\]

\( C(X) \) is UMA iff it is \( \Theta' \)-UMA with \( \Theta' = \{ \theta \}^c \).

- Intuitively, confidence sets with small probabilities of covering wrong parameter values are preferred.
- If we consider a lower confidence bound for a real-valued \( \theta \), we only need to worry about covering values of \( \theta \) that are too small, i.e., \( \Theta' = \{ \theta' \in \Theta : \theta' < \theta \} \).
Theorem 7.4

Let $C(X)$ be a confidence set for $\theta$ obtained by inverting the acceptance regions of nonrandomized tests $T_{\theta_0}$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$.

Suppose that for each $\theta_0$, $T_{\theta_0}$ is UMP of size $\alpha$.

Then $C(X)$ is $\Theta'$-UMA with confidence coefficient $1 - \alpha$, where $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$.

Proof

The fact that $C(X)$ has confidence coefficient $1 - \alpha$ follows from Theorem 7.2.

Let $C_1(X)$ be another confidence set with confidence level $1 - \alpha$.

By Proposition 7.2, the test

$$T_{1\theta_0}(X) = 1 - I_{A_1(\theta_0)}(X)$$

with $A_1(\theta_0) = \{x : \theta_0 \in C_1(x)\}$ has significance level $\alpha$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$. 
For any $\theta' \in \Theta'$, $\theta \in \Theta_{\theta'}$, i.e., $P$ is in the family defined by $H_1 : \theta \in \Theta_{\theta'}$. Thus,

\[ P(\theta' \in C(X)) = 1 - P(T_{\theta'}(X) = 1) \leq 1 - P(T_{1\theta'}(X) = 1) = P(\theta' \in C_1(X)), \]

where the first equality follows from the fact that $T_{\theta'}$ is nonrandomized and the inequality follows from the fact that $T_{\theta'}$ is UMP.

**Discussions**

Theorem 7.4 can be applied to construct UMA confidence bounds in problems where the population is in a one-parameter parametric family with monotone likelihood ratio so that UMP tests exist (Theorem 6.2). It can also be applied to a few cases to construct two-sided UMA confidence intervals.

For example, $[X(n), \alpha^{-1/n}X(n)]$ in Example 7.13 is UMA.
As we discussed in §6.2, in many problems there are UMPU tests but not UMP tests.

**Definition 7.3**

Let $\theta \in \Theta$ be an unknown parameter, $\Theta'$ be a subset of $\Theta$ that does not contain the true parameter value $\theta$, and $1 - \alpha$ be a given confidence level.

(i) A level $1 - \alpha$ confidence set $C(X)$ is said to be $\Theta'$-unbiased (unbiased when $\Theta' = \{\theta\}^c$) iff

$$P(\theta' \in C(X)) \leq 1 - \alpha$$

for all $\theta' \in \Theta'$.

(ii) Let $C(X)$ be a $\Theta'$-unbiased confidence set with confidence coefficient $1 - \alpha$. If

$$P(\theta' \in C(X)) \leq P(\theta' \in C_1(X))$$

for all $\theta' \in \Theta'$.

holds for any other $\Theta'$-unbiased confidence set $C_1(X)$ with confidence level $1 - \alpha$, then $C(X)$ is $\Theta'$-uniformly most accurate unbiased (UMAU).

$C(X)$ is UMAU if and only if it is $\Theta'$-UMAU with $\Theta' = \{\theta\}^c$. 
**Theorem 7.5**

Let $C(X)$ be a confidence set for $\theta$ obtained by inverting the acceptance regions of nonrandomized tests $T_{\theta_0}$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$.

If $T_{\theta_0}$ is unbiased of size $\alpha$ for each $\theta_0$, then $C(X)$ is $\Theta'$-unbiased with confidence coefficient $1 - \alpha$, where $\Theta' = \{ \theta' : \theta \in \Theta_{\theta'} \}$.

If $T_{\theta_0}$ is also UMPU for each $\theta_0$, then $C(X)$ is $\Theta'$-UMAU.

**Examples 7.9 and 7.15.**

Consider the normal linear model $X = N_n(Z\beta, \sigma^2 I_n)$ and the problem of constructing a confidence set for $\theta = L\beta$, where $L$ is an $s \times p$ matrix of rank $s$ and all rows of $L$ are in $\mathcal{R}(Z)$.

The LR test of size $\alpha$ for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{ X : W(X, \theta_0) \leq c_\alpha \},$$

where $c_\alpha$ is the $(1 - \alpha)$th quantile of the F-distribution $F_{s,n-r}$,

$$W(X, \theta) = \frac{[\|X - Z\hat{\beta}(\theta)\|^2 - \|X - Z\hat{\beta}\|^2]/s}{\|X - Z\hat{\beta}\|^2/(n-r)},$$
\( r \) is the rank of \( Z \), \( r \geq s \), \( \hat{\beta} \) is the LSE of \( \beta \) and, for each fixed \( \theta \), \( \hat{\beta}(\theta) \) is a solution of
\[
\|X - Z\hat{\beta}(\theta)\|^2 = \min_{\beta: L\beta = \theta} \|X - Z\beta\|^2.
\]

Inverting \( A(\theta) \), we obtain the following confidence set for \( \theta \) with confidence coefficient \( 1 - \alpha \):
\[
C(X) = \{ \theta : W(X, \theta) \leq c_\alpha \},
\]
which forms a closed ellipsoid in \( \mathbb{R}^s \).

Consider the special case of \( s = 1 \), \( \theta = l^\tau \beta \), where \( l \in \mathbb{R}(Z) \).
From §6.2.3, the nonrandomized test with acceptance region
\[
A(\theta_0) = \left\{ X : l^\tau \hat{\beta} - \theta_0 > t_{n-r, \alpha} \sqrt{l^\tau(Z^\tau Z)^{-1}lSSR/(n-r)} \right\}
\]
is UMPU with size \( \alpha \) for testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta < \theta_0 \), where \( t_{n-r, \alpha} \) is the \((1 - \alpha)\)th quantile of the t-distribution \( t_{n-r} \).

Inverting \( A(\theta) \) we obtain the following \( \Theta' \)-UMAU upper confidence bound with confidence coefficient \( 1 - \alpha \) and \( \Theta' = (\theta, \infty) \):
\[
\bar{\theta} = l^\tau \hat{\beta} - t_{n-r, \alpha} \sqrt{l^\tau(Z^\tau Z)^{-1}lSSR/(n-r)}.
\]
A UMAU confidence interval for \( \theta \) can be similarly obtained.