Discussion 4

I. Review

1. General Addition Rule:
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Additional Rule for Mutually Exclusive Events:
If events A and B are mutually exclusive,
$$P(A \cup B) = P(A) + P(B)$$

3. Probability of Complement:
$$P(A^C) = 1 - P(A)$$
   (a) The probability that an event does not happen is 1 minus the probability that it does.
   (b) P(not A)$=1 - P(A)$

4. Conditional Probability:
   (a) The conditional probability of an event given another is the probability of the event given that the other event has occurred.
   (b) If $P(B) > 0$,
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

5. Independence:
   (a) Events A and B are **independent** if information about one does not affect the other.
   (b)
$$P(A|B) = P(A) \quad \textit{or} \quad P(B|A) = P(B)$$
   (c) This is equivalent to
$$P(A \cap B) = P(A)P(B)$$

6. General Multiplication Rule:
   (a)
$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$
   (b) This rule follows directly from the definition of conditional probability.

7. The Law of Total Probability:
   (a) If B_1, B_2,... form a partition, then
$$P(A) = \sum_i P(B_i)P(A|B_i)$$
   (b) The *law of total probability* says that the unconditional P(A) is a weighted average of the conditional probabilities $P(A|B_i)$ weighted by the probabilities of the conditions $P(B_i)$.

8. Bayes Theorem:
$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

II. Practice Problems

1. A bag contains five pebbles. Three of these pebbles are black and two are white. For each of the following questions, the bag starts again in this state.

   (a) One pebble is randomly chosen from the bag. What is the probability that it is white?

   (b) One pebble is drawn from the bag and *not replaced*. The first pebble is black. What is the probability that a second randomly drawn people is white?

   (c) A pebble is drawn randomly from the bag and then returned to the bag, its color is recorded two more times (for a total of three draws). What is the probability that all three pebbles are black?

   (d) A pebble is drawn from the bag at random, its color is recorded, and then *it is not replaced in the bag*. A second pebble is drawn at random, and it is also not replaced. Finally, a third pebble is drawn from the bag. What is the probability that all three of these pebbles are black?

   (e) Why are the answers to parts (c) and (d) different?

2. An observational study investigates the connection between aspirin use and three vascular conditions – gastrointestinal bleeding, primary stroke, and cardiovascular disease – using a group of patients exhibiting these disjoint conditions with the following **prior probabilities:** $P(\text{GI bleeding}) = 0.2$, $P(\text{Stroke}) = 0.3$, and $P(\text{CVD}) = 0.5$, as well as with the following **conditional probabilities:** $P(\text{Aspirin}|\text{GI bleeding}) = 0.09$, $P(\text{Aspirin}|\text{Stroke}) = 0.04$, and $P(\text{Aspirin}|\text{CVD}) = 0.02$.

   (a) Calculate the following **posterior probabilities:** $P(\text{GI bleeding}|\text{Aspirin})$, $P(\text{Stroke}|\text{Aspirin})$, and $P(\text{Aspirin}|\text{CVD})$.

   (b) Interpret: Compare the prior probability of *each* category with its corresponding posterior probability. *What conclusion can you draw?* Be as specific as possible.

3. Down syndrome (DS) is a chromosomal condition that occurs in about one in 1000 pregnancies. One test for DS is called the triple test, which screens for levels of three hormomes in maternal blood at around 16 weeks of pregnancy. The triple test is not perfect, however. It does not always correctly identifies a fetus with DS(an error called a false negative), and sometimes it incorrectly identifies a fetus with a normal set of chromosomes as DS (an error called a false positive). Under normal conditions, the detection rate of the triple test (i.e., the probability that a fetus with DS will be correctly scored as having DS) is 0.60. The false positive rate (i.e., the probability that a test would say incorrectly that a normal fetus has DS) is 0.05. Mary did a triple test, the result is positive. What is the probability that Mary has a real DS?

III. Solutions of the Practice problems

1. (a) $2/5 = 0.4$
   (b) $2/4 = 0.5$
   (c) $\binom{5}{3}0.6^3 0.4^2 = 0.3456$
   (d) $\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = 0.1$

2. (a) $P(\text{Aspirin}) = 0.2 \times 0.09 + 0.3 \times 0.04 + 0.5 \times 0.02 = 0.04$. Then,

$$P(\text{GI}|\text{Aspirin}) = \frac{0.2 \times 0.09}{0.04} = 0.45$$
$$P(\text{Stroke}|\text{Aspirin}) = \frac{0.3 \times 0.04}{0.04} = 0.3$$
$$P(\text{CVD}|\text{Aspirin}) = \frac{0.5 \times 0.02}{0.04} = 0.25$$

   (b) Comparing the prior and posterior distribution, Aspirin can reduce the chance of having CVD and increase the chance of having GI, but has no effect on Stroke.

3. The question is to find P(DS|positive result). Before we do calculation, let us first to figure out what we have known.

   P(DS)=$\frac{1}{1000} = 0.001$, P(positive result|DS) $= 0.6$, P(positive result|no DS) $= 0.05$.

   From these we can calculate(Law of total probability),

$$P(\text{positive result}) = P(\text{positve result}|DS)P(DS) + P(\text{positive esult}|\text{no DS})P(\text{no DS})$$
$$= 0.6 * 0.001 + 0.05 * (1 - 0.001) = 0.05055$$

   Then we can use Bayes theorem

$$P(DS|\text{positive result}) = \frac{P(DS)P(\text{positve result}|DS)}{P(\text{positive result})} = \frac{0.60 * 0.001}{0.05055} = 0.012$$