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UNIFORM CONVERGENCE RATES FOR CERTAIN
APPROXIMATE SOLUTIONS TO FIRST KIND
INTEGRAL EQUATIONS

by

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ABSTRACT

This note considers convergence rates for certain approximate solutions to Fredholm integral equations of the first kind, namely

\[ u(t) = \int_{S} K(t,s) z(s) \, ds, \quad t \in T \quad (\ast) \]

where \( S, T \) are closed, bounded intervals of the real line, and \( K(t,s) \) is a given kernel on \( T \times S \) with appropriate properties. It is desired to approximate \( z(s) \) given \( u(t) \), for \( t = t_1, t_2, \ldots, t_n \). We assume that \( z \) is an element of a Reproducing Kernel Hilbert space \( \mathcal{H}_R \) with reproducing kernel \( R(s,s') \), and we let the approximate solution to \( (\ast) \) be that element \( z \in \mathcal{H}_R \) which minimizes \( \| z \|_R^2 \) subject to

\[ u(t_i) = \int_{S} K(t_i,s) z(s) \, ds, \quad i = 1, 2, \ldots, n \]

Let \( \| \Delta \| = \sup_i |t_{i+1} - t_i| \), where \( t_1 \) and \( t_n \) are the boundaries of \( T \), and let \( Q(t,t') \) be defined by

\[ Q(t,t') = \int_{S} \int_{S} K(t,s) R(s,s') K(t',s') \, ds \, ds' \]

Loosely speaking, we show that, if \( Q \) has smoothness properties similar to those of the Green's function of a \( 2m \) th order differential operator, and \( z \) is sufficiently smooth to insure that \( u \) given by \( (\ast) \) has a representation of the form

\[ u(t) = \int_{T} Q(t,t') \rho(t') \, dt' \]

for some bounded \( \rho \), then if the approximate solution to \( (\ast) \) above be denoted by \( P_{V_n} z \), we have

\[ |z(s) - (P_{V_n} z)(s)| = o \left( \| \Delta \|^m \right) \quad s \in S \]
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SOLUTIONS TO FIRST KIND INTEGRAL EQUATIONS

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1. Introduction: This note continues the study, begun in [3] of methods of solution of Fredholm Integral Equations of the first kind, namely

\[ u(t) = \int_{S} K(t, s) z(s) \, ds \quad , \quad t \in T \]  \hspace{1cm} (1.1)

where \( S, T \) are closed, bounded intervals of the real line, \( K(t, s) \) is a given kernel on \( T \times S \) with appropriate properties, and \( u(t) \) is given for \( t \in \Delta; \Delta = \{ t_1 < t_2 < \ldots < t_n, \ t_i \in T \}. \)

In [3], it was assumed that \( z \in \mathcal{H}_R \), where \( \mathcal{H}_R \) is the reproducing kernel Hilbert Space with reproducing kernel \( R(s, s') \). \( R(s, s') \) is a given positive definite kernel on \( S \times S \). Let

\[ Q(t, t') = \int_{S} \int_{S} K(t, s) R(s, s') K(t', s') \, ds \, ds'. \]  \hspace{1cm} (1.2)

It was shown in [3] that, if \( Q(t, t') \) is well defined and finite as a Riemann integral for every \( t \in T \), then the linear functional \( \Lambda_t \) on \( \mathcal{H}_R \), defined, for fixed \( t \in T \) by

\[ \Lambda_t z = \int_{S} K(t, s) z(s) \, ds \]  \hspace{1cm} (1.3)

is continuous on \( \mathcal{H}_R \) and has the representation

\[ \Lambda_t z = \langle \eta_t, z \rangle_R , \quad t \in T , \ z \in \mathcal{H}_R, \]  \hspace{1cm} (1.4)
where \( \eta_t \in \mathcal{H}_R \) is defined by

\[
\eta_t(s) = \int_S K(t, u) R(s, u) \, du
\]  \hspace{1cm} (1.5)

and \( \langle \cdot, \cdot \rangle_R \) denotes inner product in \( \mathcal{H}_R \). We will denote the subspace of \( \mathcal{H}_R \) spanned by \( \{ \eta_{t_i}, t_i \in \Delta \} \) by \( V_n \) and the projection operator in \( \mathcal{H}_R \) onto \( V_n \) by \( P_{V_n} \).

Suppose \( u(t_i) \) and \( \eta_{t_i}(s) \) are known exactly for \( t_i \in \Delta, s \in S \). Then it was proposed in [3] to choose as a "solution" to

\[
u(t_i) = \int_S K(t_i, s) z(s) \, ds , \quad t_i \in \Delta
\]  \hspace{1cm} (1.6)

that element \( z \in \mathcal{H}_R \) which minimizes \( \| z \|^2_R \) subject to the constraints (1.6). Since we may write (1.6) as

\[
u(t_i) = \langle \eta_{t_i}, z \rangle_R , \quad t_i \in \Delta.
\]  \hspace{1cm} (1.7)

the solution is \( P_{V_n} z \), and is explicitly given, in the case \( \dim V_n = n \) by

\[
(P_{V_n} z)(s) = (\eta_{t_{i_1}}(s), \eta_{t_{i_2}}(s), \ldots \eta_{t_{i_n}}(s) \overline{Q}^{-1}(u(t_{i_1}), u(t_{i_2}), \ldots u(t_{i_n}))',
\]  \hspace{1cm} (1.8)

where \( \overline{Q} \) is the \( n \times n \) Gramian of \( \{ \eta_{t_i} \}_{i=1}^n \), with \( i,j \)th entry given (see [3]) by

\[
\langle \eta_{t_{i_1}}, \eta_{t_{j_1}} \rangle_R = \overline{Q}(t_{i_1}, t_{j_1})
\]  \hspace{1cm} (1.9)

Let \( \| \Delta \| = \sup_i | t_{i+1} - t_i | \), where, without loss of generality, we let \( t_1 \) and \( t_n \) be the boundaries of \( T \).
The purpose of this note is to investigate the rate of convergence of \((P_n z)(s)\) to \(z(s)\), as \(\|\Delta\| \to 0\). Let \(R_s \in \mathcal{H}_R\) be that element of \(\mathcal{H}_R\) whose value at \(s'\) is given by
\[
R_s(s') = R(s, s')
\] (1.10)

By the properties of reproducing kernel spaces,
\[
\left< R_s, z \right>_R = z(s), \quad s \in S, \ z \in \mathcal{H}_R
\] (1.11)

and hence
\[
|z(s) - (P_n z)(s)| = |\left< z - P_n z, R_s - P_n R_s \right>_R |
\leq \|z - P_n z\|_R \|R_s - P_n R_s\|_R
\leq \|z - P_n z\|_R \|R_s\|_R
= \|z - P_n z\|_R R^\frac{1}{2}(s, s)
\] (1.12)

Suppose \(z \in \mathcal{H}_R\) and
\[
0 \equiv \int_S K(t, s) z(s) \, ds, \quad t \in T,
\] (1.13)

entails that \(z(s) \equiv 0\).

Equation (1.13) may be written
\[
0 = \left< \eta_t, z \right>_R, \quad t \in T \implies z = 0,
\] (1.14)

thus, in this case, \(\{ \eta_t, t \in T \}\) span \(\mathcal{H}_R\). \(R(s, s')\) continuous on \(S \times S\) insures that \(\mathcal{H}_R\) is separable, and, as \(\|\Delta\| \to 0\), we must have \(\|z - P_n z\|_R \to 0\), any fixed \(z\), (including \(z = R_s\)). However it appears that no rate holding uniformly for \(z \in \mathcal{H}_R\) can be found. In this note we
exhibit convergence rates for \( \| z - P_n z \|_R \) when \( z \) belongs to a special subset of \( \mathbb{H}_R \). More specifically, the main purpose of this paper is to prove

**Theorem 4.**

Let \( z \in \mathbb{H}_R \), have a representation

\[
    z(s) = \int_{T'} \eta_t(s) \rho(t) \, dt
\]

for some bounded \( \rho \), and suppose \( \{\eta_t, t \in T\} \) span \( \mathbb{H}_R \). Let \( Q(t, t') \) satisfy

1. \( \frac{\partial^\ell}{\partial t^{\ell}} Q(t, t') \) exists and is continuous on \( T \times T \) for \( t \neq t' \),

   \( \ell = 0, 1, 2, \ldots, 2m \),

   \( \frac{\partial^\ell}{\partial t^{\ell}} Q(t, t') \) exists and is continuous on \( T \times T \) for

2. \( \ell = 0, 1, 2, \ldots, 2m-2 \)

\( \lim_{t \uparrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t, t') \) and \( \lim_{t \downarrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t, t') \)

exist and are bounded for all \( t' \in T \).

Then

\[
    \| z - P_n z \|_R = O(\| \Delta \|)^m
\]

The proof, though long, rests only on an isomorphism between two appropriately chosen Hilbert spaces and the Newton form of the remainder for Lagrange Interpolation. Section 2 is devoted to the proof, which proceeds via three preliminary theorems. Section 3 is devoted to a simple example and discussion.
In the example of Section 3, \( z \in \mathcal{H}_R \) implies that \( z^{(k)} \in L_2(S) \) and \( u \) of 1.1 satisfies \( u^{(m)} \in L_2(T) \), where \( k + \ell = m \) and \( K(t, s) \) is a Green's function for an \( \ell \) th order differential operator. In that example, which is apparently typical of the general situation, the restriction (1.15) implies that \( u^{(m+j)}, j = 0, 1, 2, \ldots m-1 \) is continuous and \( u^{(2m)} \) is bounded. Thus, loosely speaking, our results apply to the situation in which \( u \) is slightly more than twice as smooth as in implied by the assumption \( z \in \mathcal{H}_R \).
Theorem 4 has an interesting

**Corollary** Let $N$ be a continuous linear functional on $\mathcal{H}_R$ with representer $\delta,$

$$Nz = \langle \delta, z \rangle_R , \quad z \in \mathcal{H}_R$$

(1.16)

and suppose that $\delta$ has the form

$$\delta(s) = \int_T \eta_t(s) \psi(t) \, dt$$

(1.17)

for some bounded $\psi.$ Let $Nz$ be approximated by $N(P_n z).$ Then

$$|Nz - N(P_n z)| = O(\|\Delta\|^m), \quad z \in \mathcal{H}_R.$$  \hspace{1cm} (1.18)

This is an immediate consequence of Theorem 4 upon noting that

$$|Nz - N(P_n z)| = \left| \left\langle z - P_n z, \delta - P_n \delta \right\rangle_R \right|$$

$$\leq \|z - P_n z\|_R \|\delta - P_n \delta\|_R$$

(1.19)

and

$$\|\delta - P_n \delta\|_R = O(\|\Delta\|^m)$$

(1.20)

2. **Proof of Theorem 4**

Theorem 4 is an immediate consequence of Theorems 1, 2, and 3 below. Theorem 1 is of independent interest and is discussed in Section 3.

**Theorem 1.**

Let $Q(t, t')$ be a positive definite kernel on $T \times T$ and let $\mathcal{H}_Q$ be
the reproducing kernel Hilbert space with $Q$ as reproducing kernel.

Suppose

\begin{equation}
\text{i) } \frac{\partial^\ell}{\partial t^\ell} Q(t, t') \text{ exists and is continuous on } T \times T \text{ for } t \neq t', \ell = 0, 1, 2, \ldots, 2m; \frac{\partial^\ell}{\partial t^{2m-1}} Q(t, t') \text{ exists and is continuous on } T \times T \text{ for } \ell \leq 2m - 2.
\end{equation}

\begin{equation}
\text{ii) } \lim_{t \uparrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t, t') \text{ and } \lim_{t \downarrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t, t')
\end{equation}

exist and are bounded for all $t' \in T$. Let $\mathcal{H}_{Q \big| T_n}$ be the subspace of $\mathcal{H}_Q$ spanned by the elements $\{Q_{t_i}\}_{i=1}^n$ where $\{t_i\}_{i=1}^n$ are $n$ distinct points in $T$ and $Q_{t_i}$ is that element of $\mathcal{H}_Q$ whose value at $t$ is given by

\begin{equation}
Q_{t_i}(t) = Q(t_i, t) \quad i = 1, 2, \ldots, n
\end{equation}

Let $P_{T_n}$ be the projection operator onto $\mathcal{H}_{Q \big| T_n}$ and $\langle \cdot, \cdot \rangle_Q$ denote inner product in $\mathcal{H}_Q$.

Let $u \in \mathcal{H}_Q$ have a representation of the form

\begin{equation}
u(t) = \int_T Q(t, t') \rho(t') \, dt'
\end{equation}

for some continuous $\rho$. Without loss of generality let $t_1$ and $t_n$ be the boundary points of $T$, and let $n = N(2m-1)$ for some integer $N$. Let $I_k$ be the interval

\begin{equation}I_k = [t_k(2m-1)+1 \ldots t_{k+1}(2m-1)+1] \quad k = 0, 1, 2, \ldots, N-1.
\end{equation}
Then
\[ \| u - P_{T_n} u \|_Q^2 \leq \sup_t \rho^2(t) (2m+1) (t_n - t_1) \times \]
\[ \left[ \sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| + 2 \sup_k \sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \right] \times \]
\[ \sum_{k=0}^{N-1} \left( t_{k+1} (2m+1) - t_k (2m+1) \right) \]
\[ = O \left( \| \Delta \|^{2m} \right) \]  
(2.5)

Remark: Since \( u(t_i) = \langle Q_{t_i}, u \rangle_Q \), \( i = 1, 2, \ldots, n \), \( P_{T_n} u \) is that element of minimum \( \mathcal{H} Q \) norm which interpolates to \( u(t_i), i = 1, 2, \ldots, n \),
\[ \langle P_{T_n} u, Q_{t_i} \rangle_Q = (P_{T_n} u)(t_i) = u(t_i) \, . \]

Proof:

If \( \tilde{u} \) is any element in \( \mathcal{H} Q \) of the form
\[ \tilde{u} = \sum_{i=1}^n Q_{t_i} \int_0^1 c_i(t) \rho(t) \, dt \]  
(2.6)
then, since \( \tilde{u} \in \mathcal{H}_{T_n} \) we have
\[ \| u - P_{T_n} u \|_Q^2 \leq \| u - \tilde{u} \|_Q^2 \]  
(2.7)

The proof proceeds by finding a set of functions \( \{ c_i(t) \}_{i=1}^n \) so that
\[ \| u - \tilde{u} \|_Q^2 \] with \( \tilde{u} \) defined by (2.6) is bounded by the right hand side of (2.5).
Now, since \( u \) satisfying (2.4) satisfies

\[
\left \langle u, u \right \rangle_Q = \int_0^1 \int_0^1 \rho(t) Q(t, t') \rho(t') \, dt \, dt'
\]

(2.8)

\[
\left \langle u, Q_{t_1} \right \rangle_Q = u(t_1) = \int_0^1 Q_{t_1}(t) \rho(t) \, dt
\]

(2.9)

it follows that

\[
\left \Vert u - \widetilde{u} \right \Vert^2_Q = \int_0^1 \int_0^1 \rho(t) \rho(t') \left \langle Q_t - \sum_{i=1}^m c_i(t) Q_{t_i}, Q_{t' - \sum_{j=1}^m c_j(t') Q_{t_j}} \right \rangle_Q \, dt \, dt'
\]

(2.10)

where \( Q_t \) is, for each fixed \( t \), that element in \( Q \) whose value at \( t' \) is given by \( Q_t(t') = Q(t, t') \). For \( t \in I_k \) we will approximate \( Q_t \) by that linear combination of \( \left \{ Q_{t_k(2m-1)+\ell} \right \}_{\ell=1}^{2m} \) which corresponds to Lagrange (polynomial) interpolation of degree \( 2m-1 \). More precisely, let

\[
p_{k, \ell}(t) = \frac{\prod_{j=1, j \neq \ell}^{2m} (t - t_k(2m-1)+j)}{\prod_{j=1}^{2m} (t_k(2m-1)+\ell - t_k(2m-1)+j)}
\]

\[=0 \quad t \notin I_k \]

t \in I_k

\( k = 0, 1, 2, \ldots, N-1 \)

\( \ell = 1, 2, \ldots, 2m \)

and let

\[
c_{k(2m-1)+\ell}(t) = p_{k, \ell}(t)
\]

\( k = 0, 1, 2, \ldots, N-1 \)

\( \ell = 1, 2, \ldots, 2m-2 \)

and

\((k, \ell) = (0, 1)\)

\[
=p_{k-1, 1}(t) + p_{k, 1}(t) \quad k = 1, 2, \ldots, n-1
\]
Then, for $f \in \mathcal{H}_Q$, $t \in I_k$,

$$
\left< Q_t - \sum_{i=1}^n c_i(t) Q_{t_i}, f \right>_Q = \left< Q_t - \sum_{k=1}^{2m} p_k(t) Q_{t_k(2m-1)\ell}, f \right>_Q
$$

$$
= f(t) - \sum_{k=1}^{2m} p_k(t) f(t_k(2m-1)\ell) \tag{2.13}
$$

where $\sum_{k=1}^{2m} p_k(t) f(t_k(2m-1)\ell)$ is, for $t \in I_k$, the $2m-1$st degree polynomial interpolating to $f$ at $t = t_k(2m-1)\ell$, $t_k(2m-1)\ell+2\ell, \ldots, t_k(2m-1)\ell+(k+1)(2m-1)\ell$.

It remains only to show that $\tilde{u}$ defined by (2.6) with $\{c_i(t)\}_{i=1}^n$ given by (2.12) has the required properties.

By the Newton form of the remainder for Lagrange interpolation, we know (see for example, Isaacson and Keller [2], p. 248) that, for $t \in I_k$

$$
f(t) = \sum_{j=1}^{2m} p_{k,j}(t) f(t_k(2m-1)\ell+j)
$$

$$
= \Pi_{j=1}^{2m} (t - t_k(2m-1)\ell+j) f[t_k(2m-1)\ell, t_k(2m-1)\ell+2, \ldots, t_k(2m-1)\ell+(k+1)(2m-1)\ell, t] \tag{2.14}
$$

where $f[t_k(2m-1)\ell, t_k(2m-1)\ell+2, \ldots, t_k(2m-1)\ell+(k+1)(2m-1)\ell, t]$ is the $2m$th order divided difference. Furthermore we know that if $f$ has $2m$ continuous derivatives in $I_k$ then

$$
f[t_k(2m-1)\ell, t_k(2m-1)\ell+2, \ldots, t_k(2m-1)\ell+(k+1)(2m-1)\ell, t] = \frac{f^{(2m)}}{(2m)!} (\xi), \tag{2.15}
$$
for some \( \xi \in I_k \). If we only know that \( f^{(2m-1)}(t) \) is continuous except for a finite number of finite jumps, then we may write the \( 2m \)th order divided difference as a divided difference of two \( 2m-1 \)st order divided differences,

\[
\frac{1}{(t_{k+1}(2m-1)+1-t_{k}(2m-1)+1)} \{ f[t_{k}(2m-1)+1, t_{k}(2m-1)+2, \ldots t_{k+1}(2m-1)+1, t] - f[t_{k}(2m-1)+2, \ldots t_{k+1}(2m-1)+1, t] \} \]  (2.16)

and know that the term in brackets in (2.16) is bounded in absolute value by \( 2 \sup_{t \in I_k} |f^{(2m-1)}(t)| \). Now let \( f_{t'} \) be that element of \( \mathcal{F}_Q \) defined by

\[
f_{t'} = Q_{t'} - \sum_{j=1}^{n} c_j(t') Q_{t'_j}, \]  (2.17)

that is,

\[
f_{t'} = Q_{t'} - \sum_{j=1}^{2m} p_{\ell,j}(t') Q_{t_{\ell}(2m-1)+j} \quad \text{for } t' \in I_{\ell} \]  (2.18)

By assumptions (i) and (ii), for fixed \( t' \in I_{\ell} \), \( f^{(2m)}_{t'}(t) \) is continuous for \( t \not\in I_{\ell} \), and, for \( t \in I_{\ell} \), \( f^{(2m-1)}_{t'}(t) \) has bounded left and right derivatives. Thus, for \( k \neq \ell \) and \( \rho = \sup_{t} |\rho(t)| \), we have
\[
\begin{align*}
&\left| \int \int p(t) p(t') \left\langle Q_t, \sum_{j=1}^{2m} p_{k,j}(t) Q_{t_k(2m-1)+j}, Q_{t_{\ell}(2m-1)+j} \right\rangle dt \, dt' \right| \\
&\leq \rho^2 \int \int \left| \left\langle Q_t, \sum_{j=1}^{2m} p_{k,j}(t) Q_{t_k(2m-1)+j}, f_{t'} \right\rangle Q \right| dt \, dt' \\
&= \rho^2 \int \int \left| \sum_{j=1}^{2m} p_{k,j}(t) \left[ f_{t', t_k(2m-1)+1}, t_{k(2m-1)+2}, \ldots, t_{k(2m-1)+j} \right] \right| dt \, dt' \\
&\leq \rho^2 \left( t_{(\ell+1)(2m-1)+1} - t_{\ell(2m-1)+1} \right) \left( t_{k(2m-1)+1} - t_{k(2m-1)+1} \right)^{2m+1} \\
&\times \left| f_{t', (2m)}(\xi) \right| \\
\end{align*}
\]

for some \( \xi \in I_k \). Since \( p_{k,j}(t) \leq 1 \), we have, for \( k \neq \ell \) that

\[
\left| f_{t', (2m)}(\xi) \right| \leq (2m+1) \sup_{\xi' \in I_k} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right|. \tag{2.20}
\]

For \( k=\ell \), by use of (2.14) and (2.16), we have

\[
\begin{align*}
&\left| \int \int p(t) p(t') \left\langle Q_t, \sum_{j=1}^{2m} p_{k,j}(t) Q_{t_k(2m-1)+j}, f_{t'} \right\rangle Q \right| dt \, dt' \\
&\leq \rho^2 \int \int \left| \sum_{j=1}^{2m} p_{k,j}(t) \left[ f_{t', t_{k(2m-1)+1}+1, \ldots, t_{k(2m-1)+2}, \ldots, t_{k(2m-1)+j} \right] \right| dt \, dt' \\
&\leq \rho^2 \left( t_{k(2m-1)+1} - t_{k(2m-1)+1} \right)^{2m+1} \left( t_{k(2m-1)+1} - t_{k(2m-1)+1} \right)^{2m+1} \sup_{\xi, t' \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right|. \tag{2.21}
\end{align*}
\]
Putting together (2.10), (2.18), (2.19), (2.20) and (2.21) gives
\[\|u - \tilde{u}\|_Q^{2} \leq \sum_{k, \ell=0}^{N-1} \int \int_{I_k \times I_{\ell}} \rho(t)\rho(t') \left( Q_{t'} - \sum_{j=1}^{2m} p_{k,j}(t)Q_{t+\sum_{j=1}^{2m} p_{k,j}(t')} \right) dt \, dt'.\]
\[\leq \rho^2(2m+1) \left[ \sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| + 2 \sup_{k} \sup_{t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \right].\]
\[\sum_{k=0}^{N-1} \left(t(k+1)(2m-1)+1 - t_k(2m-1)+1 \right)^{2m+1} = O(\| \Delta \|^{2m}) \quad (2.22)\]

**Theorem 2.**

Let \( R(s, s') \) be a positive definite kernel on \( S \times S \) with associated reproducing kernel Hilbert space \( \mathcal{H}_R \), let \( \eta(t, s) \) defined on \( T \times S \) have the property that for each fixed \( t \) the function of \( s \) defined by \( \eta_t(s) \), where
\[\eta_t(s) = \eta(t, s) \quad (2.23)\]
satisfies \( \eta_t \in \mathcal{H}_R \). Let \( Q(t, t') \) be the positive definite kernel on \( T \times T \) defined by
\[Q(t, t') = \langle \eta_t, \eta_t' \rangle_R, \quad t, t' \in T \quad (2.24)\]
and let \( \mathcal{H}_Q \) be the reproducing kernel Hilbert space with \( Q \) as reproducing kernel.

Let \( V \) be the subspace of \( \mathcal{H}_R \) spanned by the family \( \{\eta_t, \ t \in T\} \), and let \( V_n \) be the subspace of \( V \) spanned by \( \{\eta_{t_i}, \ t_i \in \Delta\} \). Let \( P_V \) and \( P_{V_n} \)
be the projection operators in $\mathcal{H}_R$ onto $V$ and $V_n$ respectively. Then

$$\| P_V z - P_{V_n} z \|_R^2 = \| u - P_{T_n} u \|_Q^2$$ (2.25)

where $u \in \mathcal{H}_Q$ is defined by

$$u(t) = \langle n_t, z \rangle_R$$ (2.26)

and $P_{T_n}$ is the projection operator in $\mathcal{H}_Q$ onto the subspace $\mathcal{H}_{T_n}$ spanned by the elements $\{ Q_{t_i}, t_i \in \Delta \}$.

**Proof:**

Since

$$\langle Q_t, Q_{t'} \rangle_Q = Q(t, t') = \langle n_t, n_{t'} \rangle_R, \quad t, t' \in T,$$ (2.27)

there is an isometric isomorphism between $\mathcal{H}_Q$ and $V$ generated by the correspondence " $\sim$ ",

$$Q_t \in \mathcal{H}_Q \sim n_t \in V \quad t \in T$$ (2.28)

Obviously

$$\mathcal{H}_{T_n} \sim V_n$$

under this is isomorphism. Furthermore, since for $z \in \mathcal{H}_R$,

$$\langle n_t, P_V z \rangle_R = \langle n_t, z \rangle_R = u(t) = \langle Q_t, u \rangle_Q,$$ (2.29)

we have

$$P_V z \sim u,$$

and

$$P_{V_n} z \sim P_{T_n} u$$ (2.30)
and hence
\[ \| P_V z - P_{V_n} z \|_R^2 = \| u - P_{T_n} u \|_Q^2 \] (2.31)

**Theorem 3.**

Let \( \rho \) be bounded on \( T \) and suppose \( z \) has a representation
\[ z(s) = \int_T \eta(t, s) \rho(t) \, dt \] (2.32)
where \( \eta(t, s) \) satisfies the hypotheses of Theorem 2.

Then \( z \in V \) and \( z \sim u \) under the correspondence "\( \sim \)" of (2.18)
where
\[ u(t) = \int_T Q(t, t') \rho(t') \, dt' \] (2.33)

**Proof:**

Let \( \Pi_\ell = \{t_{11}, t_{12}, \ldots, t_{1\ell}\} \), \( \ell = 1, 2, \ldots \) be, for each \( \ell \), a partition of \( T \), such that, for every \( t \), the Riemann sums for \( \Pi_\ell \) for the integral
\[ \int_T Q(t, t') \rho(t') \, dt' \] (2.34)
converge.

Then \( z(\ell), \ell = 1, 2, \ldots \) defined by
\[ z(\ell)(s) = \sum_{j=1}^{\ell-1} \eta_{t_j}(s) \rho(t_j \ell; (t_{j+1}, \ell - t_j \ell)) \], \( \ell = 1, 2, \ldots \) (2.35)
is a Cauchy sequence of elements in \( V \) which converge pointwise to \( z(s) \) of (2.32) and \( u(\ell), \ell = 1, 2, \ldots \) defined by
\[ u(\ell)(t) = \langle \eta_t, z(\ell) \rangle_R = \langle \eta_t, \sum_{j=1}^{\ell-1} \eta_{t_j}(s) \rho(t_j \ell; (t_{j+1}, \ell - t_j \ell)) \rangle_R \] (2.36)
\( \ell = 1, 2, \ldots \)
is a Cauchy sequence of elements in $\mathcal{H}_Q$, which converge pointwise to $u(t)$ given by (2.33). But, by (2.30), $u(\ell) \sim z(\ell)$ so we must have $u \sim z$ with $u$ and $z$ defined by (2.32) and (2.33).

**Theorem 4**

Let

$$u(t) = \int_{S} K(t,s) z(s) \, ds$$  \hspace{1cm} (2.37)

where $z$ is assumed to be in $\mathcal{H}_R$, the reproducing kernel Hilbert space with reproducing kernel $R(s,s')$. Let $Q(t,t')$ defined by

$$Q(t,t') = \int_{S} \int_{S} K(t,s) R(s,s') K(t',s') \, ds \, ds'$$  \hspace{1cm} (2.38)

be well defined as a Riemann integral and satisfy hypotheses (i) and (ii) of Theorem 1. Suppose $z \in \mathcal{H}_R$ and $u(t) \equiv 0 \Rightarrow z = 0$, and suppose $z$ has a representation of the form

$$z(s) = \int_{T} \eta(t,s) \rho(t) \, dt$$  \hspace{1cm} (2.39)

where

$$\eta(t,s) = \int_{S} K(t,s') R(s,s') \, ds'.$$  \hspace{1cm} (2.40)

Let $V_n$ be the subspace of $\mathcal{H}_R$ spanned by the elements $\{\eta_{t_1}, t_1 \in \Delta\}$ where $\eta_{t_1}$ is defined by

$$\eta_{t_1}(s) = \eta(t_1,s)$$  \hspace{1cm} (2.41)

and suppose $P_{V_n}$ is the projection operator onto $V_n$. Then
\[ |z(s) - (P/V_n z)(s)| \leq R^{\frac{1}{2}}(s, s) \cdot \sup_{t} \rho^2(t) \cdot (2m+1) \]

\[ \left[ \sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| + 2 \sup_{k} \sup_{\xi, t} I_{k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \right] \times \]

\[ \sum_{k=0}^{N-1} \left( t(k+1)(2m+1) - t(k(2m-1)+1) \right)^{2m+1} \]

\[ = O(\|\Delta\|) \]

(2.42)

**Proof**

\[ |z(s) - (P/V_n z)(s)| = | \langle z - P/V_n z, R_s \rangle_R | \]

\[ \leq \| z - P/V_n z \|_R \| R_s \|_R \]

(2.43)

Since \( u(t) = \langle \eta_t, z \rangle_R \), the assumption \( z \in \mathcal{Y}_R \), \( u(t) = 0, t \in T \Rightarrow z = 0 \)

is equivalent to the assertion \( \{ \eta_t, t \in T \} \) span \( \mathcal{Y}_R \). Then, by Theorem 3, \( z \sim u \in \mathcal{Y}_Q \) given by

\[ u(t) = \int_T Q(t, t') \rho(t') \, dt' \]

(2.44)

under the correspondence (2.28) and hence

\[ \| z - P/V_n z \|_R^2 = \| u - P/T_n u \|_Q^2 \]

(2.45)

where \( P/T_n \) is as in Theorem 1. Application of Theorem 1 to the right hand side of (2.45) then gives the result.
3. **Discussion**

In an attempt to give the reader a feel for what is going on, we discuss a very simple example. There are several interesting points which will become clear from the example. First, assume that \( \{\eta_t, \ t \in T\} \) span \( \mathcal{H}_R \) and let \( K \) be the 1:1 linear operator from \( \mathcal{H}_R \) onto \( \mathcal{H}_Q \) defined by

\[
(Kz)(t) = \int S K(t, s)z(s)ds
\]  

(3.1)

Then the "solution" \( P_{n} V_{n} z \) is given by

\[
P_{n} V_{n} z = K^{-1}(P_{n} u)
\]  

(3.2)

where \( P_{n} u \) is that element of \( \mathcal{H}_Q \) of minimum \( \mathcal{H}_Q \) norm which interpolates to \( u(t_i), i=1,2, \ldots n \). Secondly, we do not so far have the best error rate for \( |z(s) - (P_{n} V_{n} z)(s)| \) since we have not discussed the rate at which \( \|R_s - P_{n} V_{n} R_s\| \to 0 \). This is done in the example presented here. We remark that if \( R(s, s') \) and \( K(t, s) \) are appropriate Green's functions, then the theorems presented here are intimately related to some convergence theorems in the theory of spline functions. We will illustrate this remark with respect to certain polynomial splines.

Let \( S = [0, 1], \ T = [0, 1] \) and let

\[
R(s, s') = \int_0^1 \frac{(s-u)^{k-1}}{(k-1)!} (s'-u)^{k-1} (k-1)! \ du
\]  

(3.3)

and

\[
K(t, s) = \frac{(t-s)^{\ell-1}}{\ell!}
\]  

(3.4)
with \( k, \ell \) positive integers with \( k+\ell = m \). Then \( \mathcal{H}_R \) is the space of all functions \( z \) on \([0, 1]\) with \( z^{(\nu)}(0) = 0, \ \nu = 0, 1, 2, \ldots k-1, \ z^{(k)} \in L^2[0, 1], \) with
\[
\left\langle z_1, z_2 \right\rangle_R = \int_0^1 z_1^{(k)}(s) z_2^{(k)}(s) \, ds
\]
(3.5)

It follows that
\[
\eta(t, s) = \int_0^1 \frac{(t-u)^{m-1}}{(m-1)!} \frac{(s-u)^{k-1}}{(k-1)!} \, du
\]
(3.6)

\[
Q(t, t') = \int_0^1 \frac{(t-u)^{m-1}}{(m-1)!} \frac{(t'-u)^{m-1}}{(m-1)!} \, du
\]
(3.7)

\( \mathcal{H}_Q \) is thus the space of all functions \( u \) on \([0, 1]\) with \( u^{(\nu)}(0) = 0, \ \nu = 0, 1, 2, \ldots m-1, \ u^{(m)} \in L^2[0, 1], \) with
\[
\left\langle u_1, u_2 \right\rangle_Q = \int_0^1 u_1^{(m)}(s) u_2^{(m)}(s) \, ds
\]
\[
\eta^*(s, t) \text{ defined by } \eta^*(s, t) = \eta(t, s)
\]
(3.8)

is the Green's function for the operator \( L \) defined by
\[
Lz = z^{(k+m)}
\]
(3.9)

with boundary conditions
\[
z^{(\nu)}(0) = 0, \ \nu = 0, 1, 2, \ldots k-1
\]
(3.9b)
\[
z^{(\nu)}(1) = 0 \ \nu = k, k+1, \ldots k+m-1
\]

Thus, \( z \) has a representation
\[
z(s) = \int_0^1 \eta(t, s) \rho (t) \, dt
\]
(3.10)

for \( \rho \) continuous if and only if \( z^{(\nu)}(0) = 0, \ \nu = 0, 1, 2, \ldots k-1, \ z^{(\nu)}(1) = 0 \ \nu = k, k+1, \ldots k+m-1, \) and \( z^{(m+k)} = \rho \) continuous.
\[ P_{V_n} z \quad \text{given by} \]
\[ P_{V_n} z = (\eta_{t_1}, \eta_{t_2}, \ldots, \eta_{t_n}) \ Q^{-1}(u(t_1), u(t_2), \ldots, u(t_n))' \quad (1.8) \]

is the solution to the problem:

Find \( z \in \mathcal{H}_R \) to minimize

\[ \int_0^1 (z'(s))^2 \ ds \quad (3.12) \]

subject to

\[ u(t_i) = \int_0^{\frac{1}{(t_i-1)}} (t_i-s)^{\ell-1} z(s) \ ds = \int_0^{t_i} d\xi_{\ell-1} \int_0^{\xi_{\ell-1}} d\xi_{\ell-2} \ldots \int_0^{\xi_1} z(s) ds, \quad i=1, 2, \ldots, n \quad (3.13) \]

The solution to the problem: Find \( \mu \in \mathcal{H}_Q \) to minimize

\[ \int_0^1 (\mu'(m)(s))^2 \ ds \quad (3.14) \]

subject to

\[ \mu(t_i) = u(t_i), \quad i = 1, 2, \ldots, n, \quad (3.15) \]

is \( \mu = P_{T_n} u \), given by

\[ \mu = (Q_{t_1}, Q_{t_2}, \ldots, Q_{t_n}) \ Q^{-1}(u(t_1), u(t_2), \ldots, u(t_n))' \quad (3.16) \]

that is,

\[ \mu(t) = (Q(t_1, t), Q(t_2, t), \ldots, Q(t_n, t)) \ Q^{-1}(u(t_1), u(t_2), \ldots, u(t_n))' \quad (3.17) \]

Since

\[ \frac{\partial^l}{\partial s^l} Q(t_1, s) = \eta(t_1, s) \quad (3.18) \]
we have

\[ \mathcal{W}^{(l)}(s) = (P_{V_n^l} z)(s) \]  \hspace{1cm} (3.19)

\( K^{-1} u, u \in \mathcal{X}_Q \) is given by

\[ K^{-1} u = u^{(l)} \]  \hspace{1cm} (3.20)

Thus, (3.19) is an example of the relation

\[ K^{-1}(P_{T_n^l} u) = P_{V_n^l} z \]  \hspace{1cm} (3.21)

whenever

\[ Kz = u \]  \hspace{1cm} (3.22)

Hence our method of approximate solution of

\[ Kz = u \]  \hspace{1cm} (3.23)

is equivalent to interpolating \( u \) smoothly at the points \( u(t_i) \), \( i=1,2,\ldots,n \) and calculating \( K^{-1} \) of the resulting interpolating function.

In general, we have

\[ |z(s) - (P_{V_n} z)(s)| \leq \|z - P_{V_n} z\|_R \|R_s - P_{V_n} R_s\|_R \]  \hspace{1cm} (3.24)

where, by Theorem (1)

\[ \|z - P_{V_n} z\|_R^2 = \|u - P_{T_n} u\|_Q^2 \]

\[ \leq \sup_t p^2(t)(2m+1) \left[ \sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| \right] \]

\[ + 2 \sup_k \sup_{\xi, t} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \sum_{k=0}^{N-1} (t_{k+1})^{(2m-1)+1-t_{k}(2m-1)+1} \]

\[ \hspace{9cm} \]  \hspace{1cm} (3.25)
In this case
\[ \rho = z^{(m+k)} = u^{(2m)} \]  
(3.24)

\[ \sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| = 0 \]  
(3.25)

\[ \sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| = 1 \]  
(3.26)

and, since
\[ \sum_{k=0}^{N-1} (t(k+1)(2m-1)+1-t_k(2m-1)+1)^{2m+1} \leq (2m-1)^{2m} \| \Delta \|^{2m} \]

we have
\[ \| z - P_{V_n} z \|_R \leq 2(2m+1) \sup_s |z^{(m+k)}(s)| (2m-1)^m \| \Delta \|^m \]  
(3.27)

In Lemma A.1 of the Appendix, it is proved, for this case, that
\[ \| R_s - P_{V_n} R_s \|_R \leq C_{k, m}^{(m+k)}(s) \leq (2m+1) \sum_s \| z^{(m+k)}(s) | \]  
(3.28)

where \( C_{k, m} \) is given by (A.13). Hence
\[ |z(s) - (P_{V_n} z)(s)| \leq 2(2m+1) \sup_s |z^{(m+k)}(s)| \]  
(3.29)

\[ \cdot C_{k, m}^{(m+k)}(s) (2m-1)^{k-\frac{1}{2}} \| \Delta \|^{m+k-\frac{1}{2}} \]

It appears that Lemma A.1 has an easy generalization to \( R \) and \( K \) which are appropriate Green's functions, but we do not as yet have the most general conditions on \( R \) and \( K \) for which error bounds of the form \( O(\| \Delta \|)^{m+k-\frac{1}{2}} \) obtain.

*The method of proof assumes a bounded mesh ratio.
Now
\[ |z(s) - (P_n z)(s)| = |u^{(\ell)}(s) - \bar{u}^{(\ell)}(s)|, \quad \ell = 0, 1, 2, \ldots m-1. \]

\[ \leq 2(2m+1) \sup_s |u^{(2m)}(s)| C_{k, m} (m-1)^{k-\frac{1}{2}} \| \Delta \|^{2m-\ell-\frac{1}{2}} (3.30) \]

Since \( \bar{u}(s) \) is the solution of the minimization problem of (3.14) and (3.15), it is well known that \( \bar{u}(s) \) is the (unique) 2m-1st degree polynomial spline of interpolation to \( u(t_i), t = 1, 2, \ldots, n \), satisfying the boundary conditions

\[ \bar{u}^{(\nu)}(0) = 0, \quad \nu = 0, 1, 2, \ldots m-1 \]

We have thus proved in (3.30) the following Theorem, which is typical of convergence theorems to be found in the spline literature. (see e.g. [1])

**Theorem 5.**

Let \( u(t) \in C^{2m}[0, 1] \), with \( u^{(\nu)}(0) = 0, \quad \nu = 0, 1, 2, \ldots m-1 \), \( u^{(\nu)}(1) = 0, \quad \nu = m, m+1, \ldots 2m-1 \), and let \( \bar{u}(t) \) be the 2m-1st degree polynomial spline of interpolation to \( u(t) \) at \( t = t_1, t_2, \ldots t_n \) satisfying the boundary conditions \( \bar{u}^{(\nu)}(0) = 0, \quad \nu = 0, 1, 2, \ldots m-1 \). Then

\[ |u^{(\ell)}(s) - \bar{u}^{(\ell)}(s)| \leq 2(2m+1) \sup_s |u^{(2m)}(s)| C_{k, m} (m-1)^{k-\frac{1}{2}} \| \Delta \|^{2m-\ell-\frac{1}{2}} \]

(3.31)

\( \ell = 1, 2, \ldots m-1. \)
Appendix.

This appendix is devoted to the proof of Lemma A.1 used in the Example of Section 3. The Lemma is not necessarily new, but is typical of convergence theorems in the spline literature, see e.g. [1].

Lemma A.1

Let
\[ R(s, s') = \int_0^1 \frac{(s-u)^{k-1}}{(k-1)!} \frac{(s'-u)^{k-1}}{(k-1)!} \, du \]
and
\[ K(t, s) = \frac{(t-s)^{k-1}}{(t-1)!}, \quad \text{with } k + \ell = m. \]
Let
\[ \sup_i |t_{i+1} - t_i|/\inf_i |t_{i+1} - t_i| = a < \infty \] (A.1)

Then
\[ \| R_s - P_n R_s \|_R \leq C_{k, m} m^{-\frac{1}{2}} \| \Delta \|^{k-\frac{1}{2}} \] (A.2)

where
\[ C_{k, m} = \left[ \frac{1}{(k-1)!} + \sum_{\tau=1}^{m-1} \frac{1}{(m-1)!} \frac{(\tau\alpha)^{m-1}}{\Pi_{\ell \neq \tau} |\tau - \ell|} \left( \frac{m-1}{\ell} \right) \right] \]
\[ \ell = 0, 1, 2, \ldots, m-1 \]

Proof.

\( \eta_t(s) \) is given by
\[ \eta_t(s) = \int_0^1 \frac{(t-u)^{m-1}}{(m-1)!} \frac{(s-u)^{k-1}}{(k-1)!} \, du \] (A.3)
For the remainder of the proof we consider \( s \) fixed. Then for any \( \{d_i(s)\}_{i=1}^{n} \),

\[
\left\| R_s - P_{V_n} R_s \right\|_2^2 \leq \left\| R_s - \sum_{i=1}^{n} d_i(s) \eta_{i} \right\|_R^2
\]

\[= \int_{0}^{1} \frac{(s-u)_{+}^{k-1}}{(k-1)!} \frac{(s-u)_{+}^{k-1}}{(k-1)!} \, du - 2 \sum_{i=1}^{n} d_i(s) \int_{0}^{1} \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \frac{(s-u)_{+}^{k-1}}{(k-1)!} \, du
\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(s) d_j(s) \int_{0}^{1} \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \frac{(t_j-u)_{+}^{m-1}}{(m-1)!} \, du
\]

\[= \int_{0}^{1} \left( \frac{(s-u)_{+}^{k-1}}{(k-1)!} - \sum_{i=1}^{n} d_i(s) \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \right)^2 \, du \quad (A.5)
\]

Suppose \( t_j \leq s \leq t_{j+1} \). Set \( d_i(s) = 0 \), except for \( i = j, j+1, \ldots \)

\( j+m-1 \). Then (A.5) can be written

\[
\int_{0}^{t_j} \left( \frac{(s-u)_{+}^{k-1}}{(k-1)!} - \sum_{i=j}^{j+m-1} d_i(s) \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \right)^2 \, du + \int_{t_j}^{t_{j+m-1}} \left( \frac{(s-u)_{+}^{k-1}}{(k-1)!} - \sum_{i=j}^{j+m-1} d_i(s) \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \right)^2 \, du
\]

\[= \int_{0}^{t_j} \left( \frac{(s-u)_{+}^{k-1}}{(k-1)!} - \sum_{i=j}^{j+m-1} d_i(s) \frac{(t_i-u)_{+}^{m-1}}{(m-1)!} \right)^2 \, du \quad (A.6)
\]

The first term may be made to vanish by choosing the \( \{d_i(s)\}_{i=j}^{j+m-1} \)

so that the coefficient of \( u^\ell \) is 0, for \( \ell = 0, 1, 2, \ldots m-1 \).
This is accomplished by letting \( \{d_i(s)\}_{i=j}^{j+m-1} \) satisfy

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
t_j & t_{j+1} & \cdots & t_{j+m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{t_j^{m-1}}{(m-1)!} & \frac{t_{j+1}^{m-1}}{(m-1)!} & \cdots & \frac{t_{j+m-1}^{m-1}}{(m-1)!}
\end{pmatrix}
\begin{pmatrix}
d_j(s) \\
d_{j+1}(s) \\
\vdots \\
d_{j+m-1}(s)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
1 \\
\frac{s^2}{2!} \\
\vdots \\
\frac{s^{k-1}}{(k-1)!}
\end{pmatrix}
\quad (A. 7)
\]

We may get an explicit expression for \( d_{j+\tau}(s), \quad \tau = 0, 1, 2, \ldots m-1 \), by noting that \( c_{j+\tau}(s), \quad \tau = 0, 1, 2, \ldots m-1 \) defined by

\[
\begin{pmatrix}
c_j(s) \\
c_{j+1}(s) \\
\vdots \\
c_{j+m-1}(s)
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & \cdots & 1 \\
t_j & t_{j+1} & \cdots & t_{j+m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{t_j^{m-1}}{(m-1)!} & \frac{t_{j+1}^{m-1}}{(m-1)!} & \cdots & \frac{t_{j+m-1}^{m-1}}{(m-1)!}
\end{pmatrix}
\begin{pmatrix}
1 \\
s \\
\frac{s^2}{2!} \\
\vdots \\
\frac{s^{m-1}}{(m-1)!}
\end{pmatrix}
\quad (A. 8)
\]

is the \((m-1)\)st degree polynomial which is \(1\) at \(s = t_{j+\tau}\) and is \(0\) at \(s = t_{j+\ell}, \quad \ell = 0, 1, 2, \ldots m-1, \quad \ell \neq \tau\). Thus

\[
c_{j+\tau}(s) = \frac{\prod_{\ell \neq \tau} (s - t_{j+\ell})}{\prod_{\ell \neq \tau} (t_{j+\tau} - t_{j+\ell})}, \quad \tau = 0, 1, 2, \ldots m-1
\quad (A. 9)
\]
and
\[ d_{j+\tau}(s) = \frac{d}{ds} \ell \quad c_{j+\tau}(s) = \frac{1}{\prod_{\ell \neq \tau} (t_{j+\tau} - t_{j+\ell})} \sum_{\ell=0,1,2,\ldots,m-1} \prod_{i_1, i_2, \ldots, i_\ell} (s-t_{j+\nu}) \]
where the sum is over all \( \binom{m-1}{\ell} \) ways the indices \( \{0,1,2,\ldots,\tau-1,\tau+1,\ldots,m-1\} \) may be selected \( \ell \) at a time. With this choice of \( \{d_{j+\tau}(s)\}_{\tau=0}^{m-1} \)

(A.6) becomes
\[
\frac{t_{j+m-1}}{t_j} \left[ \frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{\tau=1}^{m-1} \frac{(t_{j+\tau} - u)_+^{m-1}}{(m-1)!} \right] \frac{1}{\prod_{\ell \neq \tau} (t_{j+\tau} - t_{j+\ell})} \sum_{\ell=0,1,2,\ldots,m-1} \prod_{i_1, i_2, \ldots, i_\ell} (s-t_{j+\nu}) \right] du
\]

Since \( \left| (t_{j+\tau} - u)_+ / (t_{j+\tau} - t_{j+\ell}) \right| \leq \frac{\tau}{|\tau-\ell|} \) a and \( t_j \leq s \leq t_{j+1} \),

we have that (A.11) is certainly bounded by
\[
C_{k,m}^2 (m-1)^{2k-1} \| \Delta \|^{2k-1}. \quad (A.12)
\]

where
\[
C_{k,m} = \left[ \frac{1}{(k-1)!} + \sum_{\tau=1}^{m-1} \frac{1}{(m-1)!} \frac{(\tau a \cdot m-1)}{\prod_{\ell \neq \tau} (|\tau-\ell|)} \right] \quad (A.13)
\]

and
\[
\| R_S - P_{V_n} R_S \| \_R = \mathcal{O}( \| \Delta \|^{k-\frac{1}{2}} ) \quad (A.14)
\]

Remark. The case of general \( R \) may be converted to a problem in the rate of convergence of an approximation in the \( L_2 \) norm as follows. Assume \( R \) has a factorization of the form
\[ R(s, s') = \int_U G(s, u) G(s', u) \, du \]  \hfill (A.15)

where \( U \) is, say, a given interval, and assume that \( G \) and \( K \) are sufficiently regular to allow interchanges of the order of integration below. Observe that

\[
\langle R_s, R_s \rangle_R = R(s, s) = \int_U G^2(s, u) \, du
\]

\[
\langle R_s, \eta_{t_i} \rangle_R = \eta_{t_i}^2(s) = \int_S K(t_i, s') R(s, s') ds'
\]

\[
= \int_U du \, G(s, u) \int_S K(t_i, v) \, G(v, u) \, dv \hfill (A.17)
\]

and

\[
\langle \eta_{t_i}, \eta_{t_j} \rangle_R = \int_S \int_S K(t_i, s) R(s, s') K(t_j, s') \, ds \, ds'
\]

\[
= \int_U du \int_S K(t_i, v) G(v, u) \, dv \int_S K(t_j, v') G(v', u) \, dv' \hfill (A.18)
\]

Then, for any \( \{d_i(s)\}_{i=1}^n \),

\[
\| R_s - P_{V_n} R_s \|_R^2 \leq \| R_s - \sum_{i=1}^n d_i(s) \eta_{t_i} \|_R^2
\]

\[
= \int_U \left[ G(s, u) - \sum_{i=1}^n d_i(s) \int_S K(t_i, v) \, G(v, u) \, dv \right]^2 \, du \hfill (A.16)
\]

The problem of finding error bounds is then one of finding \( d_i(s) \) so that

\[
\left\{ \int_S K(t_i, v) G(v, u) \, dv \right\}_{i=1}^n \text{ well approximate } G(s, \cdot) \text{ in the } L_2 \text{ norm.}
\]
References

