Some Tests of Independence for Stationary Multivariate Time Series

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SUMMARY

Let

\[ X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_P(t) \end{pmatrix}, \quad t = \ldots -1, 0, 1, \ldots \]

be a \( P \)-dimensional zero-mean stationary Gaussian time series possessing a spectral density matrix \( F(\omega) = \{ f_{ij}(\omega) \}, i, j = 1, 2, \ldots, P \), satisfying some mild regularity conditions. Let \( \hat{F}(\omega_l), l = 1, 2, \ldots, M, \) \( \tilde{F}(\omega_l) = \{ \hat{f}_{ij}(\omega_l) \} \) be suitably defined sample spectral density matrices for \( M \) values of \( \omega_l \) based on a record of length \( T > M \). We consider the following (null) hypotheses:

\[ H_1: \quad X_i(t), X_j(s) \text{ independent if } i \neq j, \forall s, t, \]

\[ H_2: \quad X_i(t) \text{ independent of } X_j(s), j = 2, \ldots, P, \forall s, t, \]

\[ H_3: \quad X_i(t), X_j(t+\tau) \text{ independent, all } \tau \neq 0 \text{ ("white noise"}). \]

Approximate likelihood-ratio tests are found and the test statistics \( \lambda_i, \quad i = 1, 2, 3, \) are found to be functions of the \( \{ \hat{F}(\omega) \}, i = 1, 2, \ldots, M \). The \( \{ \hat{F}(\omega) \} \) converge in mean square to a family of independent complex Wishart matrices. As a consequence of this fact, \( \log \lambda_1 \) and \( \log \lambda_3 \) converge in first mean to random variables whose null densities can be given explicitly, being distributed as the logarithms of products of independent beta random variables with integer indices. Under the null hypothesis \( \lambda_3 \) is distributed as \( \tilde{\lambda}_3 \) where \( \tilde{\lambda}_3 \) is Bartlett's statistic for homogeneity of variances. The asymptotic distributions of \( \log \lambda_i, \quad i = 1, 2, 3, \) under the alternative are discussed. Under the alternative, \( \log \lambda_3 \) tends in first mean to \( \log \tilde{\lambda}_3 \). A power series expansion for the characteristic function of \( \log \lambda_3 \) under the alternative is given.

1. INTRODUCTION

Let

\[ X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_P(t) \end{pmatrix}, \quad t = \ldots -1, 0, 1, \ldots \]

be a \( P \)-dimensional zero mean stationary Gaussian time series, possessing a strictly positive definite spectral density matrix \( F(\omega) \) satisfying regularity conditions to be stated. The observed data consist of a single record of length \( T \).
In this note we consider approximate likelihood-ratio tests for the three null hypotheses given in the summary. These hypotheses are equivalently stated as

\[
H_1: \quad F(\omega) \text{ diagonal, all } \omega,
\]
\[
H_2: \quad f_{ij}(\omega) = 0, \quad j = 2, \ldots, P, \quad \text{all } \omega,
\]
\[
H_3: \quad f_{21}(\omega) = \text{constant},
\]
where

\[
F(\omega) = \{f_{ij}(\omega)\}, \quad \text{a } P \times P \text{ matrix}, \quad (1.1)
\]
\[
f_{ij}(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \exp(-i\omega\tau) R_{ij}(\tau), \quad (1.2)
\]
\[
R_{ij}(\tau) = E[X_i(t)X_j(t+\tau)]. \quad (1.3)
\]

First observe that, in the case of \(H_1\) and \(H_2\), values of \(PT\) random variables are observed, whereas the hypotheses involve a countably infinite number of parameters. In fact, the number of parameters involved in the joint density of \(X(t), t = 1, 2, \ldots, T\), is \(PT + (2T - 1)P(P - 1)/2\). To have the problem make sense, we must then assume that some of the parameters are negligible. In this paper we always assume

\[
\text{Condition A} \quad \sum_{\tau=-\infty}^{\infty} \sum_{\lambda=1}^{P} |R_{\lambda}(\tau)| = 0 < \infty. \quad (1.4)
\]

This condition ensures that the entries \(f_{ij}(\omega)\) of the spectral density matrix exist and have derivatives bounded by \(\theta\), and that

\[
\Lambda(\omega) \leq \Lambda/2\pi < \infty, \quad \text{all } \omega, \quad (1.5a)
\]

where \(\Lambda(\omega)\) is the largest eigenvalue of \(F(\omega)\). We also assume

\[
\text{Condition B} \quad \lambda(\omega) > \lambda/2\pi > 0, \quad \text{all } \omega, \quad (1.5b)
\]

where \(\lambda(\omega)\) is the smallest eigenvalue of \(F(\omega)\). Condition B guarantees that the process is non-degenerate.

We now define the test statistics. Let

\[
I_{\mu,v}(\omega) = \frac{1}{2\pi T} \sum_{s=1}^{T} \sum_{i=1}^{T} X_i(s) \exp\{is(2\pi\omega/T)\} X_j(t) \exp\{-it(2\pi\omega/T)\},
\]

\[
\mu, \nu = 1, 2, \ldots, P, \quad j = 1, 2, \ldots, T. \quad (1.6)
\]

Without appreciable loss of generality, suppose \(T\) is odd and \(n\), \(M\) and \(T\) are integers satisfying \((2n+1)M = (T-1)/2\). The desired relationship between \(n\) and \(M\) (both large) will become evident later. Let \(j_i = (i-1)(2n+1)+(n+1), \quad i = 1, 2, \ldots, M\), and let \(\omega_i = 2\pi j_i/T\). Define \(f_{\mu,v}(\omega_i), \quad l = 1, 2, \ldots, M\), by

\[
f_{\mu,v}(\omega_i) = \frac{1}{2\pi T} \sum_{s=1}^{T} \sum_{t=n}^{T} I_{\mu,v}(\omega_i + 2\pi j_i/T), \quad l = 1, 2, \ldots, M, \quad (1.7)
\]

and the \(P \times P\) matrices \(\hat{F}(\omega), \quad l = 1, 2, \ldots, M\), by

\[
\hat{F}(\omega) = \{f_{\mu,v}(\omega_i)\}. \quad (1.8)
\]

\(\hat{F}(\omega)\) is a sample spectral density matrix formed from averages of periodograms. All of the spectral quantities discussed depend on \(n\) and \(M\), we therefore suppress this in the notation.

We choose \(\lambda_1, \lambda_2\) and \(\lambda_3\) respectively.

\[
\lambda_1 = \lambda_2 = \lambda_3 = \lambda
\]

where \(\hat{F}(\omega)\) is the \(P-1 \times P-1\) matrix.

We now discuss the matrix \(F(\omega), \quad l = 1, 2, \ldots, M, \quad (\text{1.1})\), with complex Wishart distribution

\[
F(\omega) \sim W_{\lambda}(\nu, \nu - 1)
\]

By the smoothness properties

\[
\max_{\frac{|\xi|}{\sqrt{n}(\sqrt{2\pi n})^{2np}}} \max_{|\xi| < \sqrt{n}(\sqrt{2\pi n})^{2np}}
\]

where \(f_{\mu,v}(\omega_i)\) is the \(\mu\)th entry

Let \(\hat{F}(\omega), \quad l = 1, 2, \ldots, M, \quad (\text{1.1})\), with complex Wishart distribution

\[
\hat{F}(\omega) \sim W_{\lambda}(\nu, \nu - 1)
\]

The complex Wishart distribution random matrices \(\hat{F}(\omega), \quad l = 1, \ldots, M\), as the random matrices \(\hat{F}(\omega), \ldots, \hat{F}(\omega), \ldots, \hat{F}(\omega)\), likelihood-ratio statistics base

\[
\hat{H}_1: \quad F(\omega)
\]
\[
\hat{H}_2: \quad f_{\mu,v}(\omega)
\]
\[
\hat{H}_3: \quad f_{\mu,v}(\omega)
\]

are monotone functions of \(\lambda_1\), \(\lambda_2\), \(\lambda_3\)

\[
\lambda_1 = \lambda_2 = \lambda_3
\]

where \(f_{\mu,v}(\omega)\) is the \(\mu\)th entry

Elements are \(f_{\mu,v}(\omega), \mu, v = 2, \ldots, P\).
We choose \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) given below as statistics for testing \( H_1, H_2 \) and \( H_3 \) respectively.

\[
\lambda_1 = \prod_{r=1}^{M} \left| \mathbf{F}(\omega_1) \right| \prod_{\nu=1}^{P} \mathbf{J}_{\nu}(\omega_1),
\]

(1.9a)

\[
\lambda_2 = \prod_{l=1}^{M} \left| \mathbf{F}(\omega_l) \right| \left| \mathbf{F}_{22}(\omega_l) \right|,
\]

(1.9b)

where \( \mathbf{F}_{22}(\omega_l) \) is the \( P-1 \times P-1 \) matrix whose elements are \( f_{\mu \nu}(\omega_l), \mu, \nu = 2, \ldots, P, \)

\[
\lambda_3 = \prod_{l=1}^{M} \mathbf{J}_{11}(\omega_l) \left( \frac{1}{M} \sum_{l=1}^{M} \mathbf{J}_{11}(\omega_l) \right)^{M}.
\]

(1.9c)

We now discuss the motivation for these choices. Define the \( P \times P \) matrices

\[
\mathbf{F}(\omega_l), l = 1, 2, \ldots, M,
\]

by

\[
\mathbf{F}(\omega_l) = \frac{1}{2n+1} \sum_{j=-n}^{n} \mathbf{F}(\omega_l + 2\pi j/T), \quad l = 1, 2, \ldots, M.
\]

(1.10)

By the smoothness properties of \( \mathbf{F}(\omega) \) implied by (1.4), we have

\[
\max_{|\xi| \leq 2\theta(2n+1)/T} |f_{\mu \nu}(\omega_l) - f_{\mu \nu}(\omega_l + \xi)| \leq 2\theta(2n+1)/T,
\]

(1.11)

where \( f_{\mu \nu}(\omega_l) \) is the \( \mu \times \nu \)th entry of \( \mathbf{F}(\omega_l) \).

Let \( \mathbf{F}(\omega_l), l = 1, 2, \ldots, M, \) be \( M \) random \( P \times P \) matrices independently distributed with complex Wishart distributions with \( 2n+1 \) degrees of freedom,

\[
\mathbf{F}(\omega_l) \sim W_c(\mathbf{F}(\omega_0), P, 2n+1), \quad l = 1, 2, \ldots, M.
\]

(1.12)

The complex Wishart distribution is described in more detail in Section 2. The random matrices \( \mathbf{F}(\omega_l), l = 1, 2, \ldots, M, \) are, for large \( n, M \) approximately distributed, as the random matrices \( \mathbf{F}(\omega_l), l = 1, 2, \ldots, M, \) in a sense to be made precise later. The likelihood-ratio statistics based on \( \mathbf{F}(\omega_l), l = 1, 2, \ldots, M, \) for the three hypotheses

\[
\mathbf{H}_1: \quad \mathbf{F}(\omega_l) \text{ diagonal, } l = 1, 2, \ldots, M,
\]

\[
\mathbf{H}_2: \quad f_{1j}(\omega_l) = 0, j = 2, \ldots, P, l = 1, 2, \ldots, M,
\]

\[
\mathbf{H}_3: \quad f_{11}(\omega_l) = \text{const, } l = 1, 2, \ldots, M,
\]

are monotone functions of \( \lambda_1, \lambda_2, \lambda_3 \) given by

\[
\lambda_1 = \prod_{l=1}^{M} \left| \mathbf{F}(\omega_l) \right| \prod_{\nu=1}^{P} \mathbf{J}_{\nu}(\omega_l),
\]

(1.13a)

\[
\lambda_2 = \prod_{l=1}^{M} \left| \mathbf{F}(\omega_l) \right| \left| \mathbf{F}_{22}(\omega_l) \right|,
\]

(1.13b)

where \( f_{\mu \nu}(\omega_l) \) is the \( \mu \times \nu \)th entry in \( \mathbf{F}(\omega_l) \) and \( \mathbf{F}_{22}(\omega_l) \) is the \( P-1 \times P-1 \) matrix whose elements are \( f_{\mu \nu}(\omega_l), \mu, \nu = 2, 3, \ldots, P, \)

\[
\lambda_3 = \prod_{l=1}^{M} \mathbf{J}_{11}(\omega_l) \left( \frac{1}{M} \sum_{l=1}^{M} \mathbf{J}_{11}(\omega_l) \right)^{M}.
\]

(1.13c)
In this paper we discuss the distribution theory for \( \log \lambda_i \), \( i = 1, 2, 3 \). The theory is useful here because \( \log \lambda_i \) converges to a statistic distributed as \( \log \lambda_i \), for \( i = 1, 2, 3 \). The convergence theorems and proofs are to be found in Wahba (1968); the theorems are restated in Section 2.

Each term in the product in \( \lambda \) and \( \bar{\lambda} \) has a well-known real multivariate analogue, as follows. Suppose \( X = (X_1, X_2, ..., X_P) \) is a normal \( P \)-dimensional random vector with mean 0 and \( P \times P \) covariance matrix \( S \) and let \( \tilde{S} = \{\tilde{s}_{ij}\} \) be the sample covariance matrix based on \( n \) independent observations on \( X \). Then the likelihood-ratio statistic for \( A_1 : S \) diagonal is a monotone function of

\[
\eta_1 = \left| \frac{1}{P} \prod_{j=1}^{P} \tilde{s}_{jj} \right| \tag{1.14a}
\]

and the likelihood ratio statistic for \( A_2 : s_{ij} = 0, j = 2, 3, ..., P, \) is a monotone function of

\[
\eta_2 = \left( \frac{1}{s_{11}} \frac{1}{s_{22}} \right) \tilde{S} = \begin{pmatrix}
\tilde{s}_{11} & \tilde{s}_{12} \\
\tilde{s}_{21} & \tilde{s}_{22}
\end{pmatrix} \tag{1.14b}
\]

(see, for example, Anderson, 1958). This similarity is not surprising, since \( F(\omega) \) and \( S \) share a number of theoretical properties (see Koopmans, 1965). Under \( A_1 \) and \( A_2 \), these statistics are well known to be distributed as products of independent beta random variables, with, in general, non-integer indices (Anderson, 1958); numerous authors (for example, Consul, 1965, and Khatri, 1965) have discussed methods for approximating the densities. In going from the real to the complex case, we generally find that the number of degrees of freedom is doubled; here, each term in the product in (1.13a) and (1.13b) may be shown to be distributed as the product of independent beta random variables with integer indices. A simple procedure is used to exhibit the exact null moments and densities of \( \log \lambda_i \) and \( \log \bar{\lambda}_i \) and the asymptotic means and variances (large \( n \)) of \( \log \lambda_i \) and \( \log \bar{\lambda}_i \) under the alternative are given. \( \bar{\lambda}_i \) is distributed as Bartlett's statistic for homogeneity of variances, with the appropriate choices of degrees of freedom. This statistic has been discussed by numerous authors. Under the null hypothesis \( \lambda_i \) is distributed (in general) as a product of independent beta random variables with non-integer indices. Wilks (1938) gave the moments of \( \lambda_i - 1 \) under the alternative, Whittle (1952) discussed \( \lambda_i \) in the context of \( \mathbb{H}_0 \) and gave the null characteristic function and cumulants for \( \log \lambda_i \) with \( n = 1 \). Very little seems to be known about the form of the alternative density. The characteristic function of \( \log \lambda_i \) under the alternative is here expressed as an infinite weighted sum of characteristic functions. A nearby alternative may be defined as one for which \( \text{var} f_{11}/f_{11}^2 \) is small, where

\[
\begin{align*}
\text{var} f_{11} &= \frac{1}{2\pi} \int_0^{2\pi} \{f_{11}(\omega) - f_{11}(\omega)^2\} d\omega, \\
f_{11} &= \frac{1}{2\pi} \int_0^{2\pi} f_{11}(\omega) d\omega.
\end{align*}
\tag{1.15}
\]

For a nearby alternative the first few terms in the expression for the characteristic function of \( \log \lambda_i \) should suffice. An approximate expression for the first two moments of \( \log \lambda_i \) for nearby alternatives is given.


2. APPROXIMATION OF THE DISTRIBUTION OF $\log \lambda_i$ BY THAT OF $\log \bar{\lambda}_i$

Restating ideas introduced by Goodman (1963) we first describe a complex normal random vector and a complex Wishart random matrix.

$$Z = (Z_1, Z_2, ..., Z_P)' = U + iV$$

is said to be a $P$-dimensional complex normal random vector (with zero mean) if $U = (U_1, U_2, ..., U_P)'$ and $V = (V_1, V_2, ..., V_P)'$ are two real $P$-dimensional normal random vectors with the following special covariance structure

$$
\begin{bmatrix}
EU_j U_k & EU_j V_k \\
EU_k V_j & EV_k V_k
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}

= \begin{bmatrix}
f_{jk} & \text{if } j = k = 1, 2, ..., P, \\
0 & \end{bmatrix}
\begin{bmatrix}
c_{jk} & -q_{jk} \\
q_{jk} & c_{jk}
\end{bmatrix}
\begin{bmatrix}
1 & \text{if } j \neq k.
\end{bmatrix}
$$

(2.1)

It is shown in Goodman (1963) that

$$\mathbb{E}ZZ^* = F,$$

where $F$ is the hermitian matrix with $f_{jk} = \text{entry } j \neq k,$ and $f_{jk} = c_{jk} + iq_{jk}, j < k \quad (q_{jk} = -q_{kj}),$ and that the density function for the $2P$ random variables $Z$ may be written

$$p(Z) = \frac{1}{\pi^P |F|} \exp(-Z'F^{-1}Z^*).$$

(2.3)

Given $2n + 1$ independent samples $Z_r, r = 1, 2, ..., 2n + 1$, from the density $p(Z)$, the maximum-likelihood estimate $\hat{F}$ for $F$ is

$$\hat{F} = \frac{1}{2n + 1} \sum_{r=1}^{2n+1} Z_r Z_r^*.$$  

(2.4)

$\hat{F}$ is a sufficient statistic for $F$ and has the complex Wishart distribution, denoted $W_c(F, P, 2n + 1)$, with density

$$p(\hat{F}) = \frac{1}{\Gamma(P + 2n + 1)} \prod_{l=1}^{P} \Gamma(P + 2n + 1) \times |F|^{P+2n+1-1} \times |\hat{F}|^{2n+1} \exp(-tr \hat{F}^{-1} F).$$

(2.5)

Given the independent complex Wishart random matrices $\hat{F}(\omega_l), l = 1, 2, ..., M$, of (1.12), the assertions in the Introduction concerning the likelihood-ratio statistics for $\bar{H}_i$, $\bar{H}_g$ and $\bar{H}_a$ are easily demonstrated following the differentiation argument of Khatri (1965, Section 3).

The assertion that $\hat{F}(\omega_l), l = 1, 2, ..., M$, for large $n, M$ are approximately distributed as $\hat{F}(\omega_l), l = 1, 2, ..., M$, is justified by the following Theorem 1. The proof for $P = 2$ is to be found in Wahba (1968, Section 4) and extends directly to general $P$.

Theorem 1. Let conditions A and B be satisfied. Then, for each $n, M$ it is possible to construct, on the same sample space as $X(t), i = 1, 2, ..., T$, a family $\hat{F}(\omega_l), l = 1, 2, ..., M$, of independent random complex Wishart matrices,

$$\hat{F}(\omega_l) \sim W_c(\hat{F}(\omega_l), P, 2n + 1), \quad l = 1, 2, ..., M.$$
such that
\[
E \sum_{i=1}^{M} \text{trace} \left( \hat{F}(\omega_i) - \bar{F}(\omega_i) \right) \left( \hat{F}(\omega_i) - \bar{F}(\omega_i) \right)^* \leq 6P(\Lambda/\lambda)^2 (2n+1) M/T^2 + 8P (1 + 2\Lambda/\lambda)^2 \theta^2 \log_2 M/((2n+1)^2).
\]

We thus have simultaneous quadratic mean convergence as \( T \to \infty \), if, for example, \((\log M)/n^2 \to 0\).

Henceforth, we let \( \bar{\lambda}_i, i = 1, 2, 3 \), be functions of the complex Wishart matrices \( \hat{F}(\omega_i), \omega_i = 1, 2, \ldots, M \), defined by (1.13a), (1.13b) and (1.13c). It will be shown in Section 3 that, under the general alternative, \( F(\omega) \) general, conditions A and B satisfied, that \( \text{var} \left( \log \lambda_i \right) = O(M/n) \), \( i = 1, 2, 3 \). It will be shown that, under \( H_0 \), \( \text{var} \left( \log \lambda_i \right) = O(M/n^2) \), \( i = 1, 2 \). Under \( H_3 \), we may, without loss of generality, take \( P = 1 \). Let \( f_1(\omega) = f \). In this case it is well known that \( \{2(2n+1)f_2(\omega_i)/f_i, i = 1, 2, \ldots, M \} \) are independently distributed as \( \chi^2_{2(2n+1)} \) and that \( \{2(2n+1)f_3(\omega_i)/f_i, i = 1, 2, \ldots, M \} \) are also independently distributed as \( \chi^2_{2(2n+1)} \), so that under \( H_3 \), \( \lambda_3 \) and \( \lambda_{\bar{3}} \) have the same distribution.

We may approximate the distribution of \( \log \lambda_i \) by that of \( \log \bar{\lambda}_i, i = 1, 2, 3 \), because of the following.

**Theorem 2.** Let conditions A and B be satisfied. Then, as \( n, M \to \infty \) in such a way that \( (\log_2 M)/n \to 0 \)

\[
E(n/M)^i |\log \lambda_i - \log \bar{\lambda}_i| = O \left( \frac{\log M}{n} \right), \quad i = 1, 2, 3.
\]

If \( H_0, i = 1, 2, \) are true, then

\[
E(n/M^i) |\log \lambda_i - \log \bar{\lambda}_i| = O \left( \frac{\log M}{n} \right), \quad i = 1, 2.
\]

A proof for \( i = 1, 2, P = 2 \) is to be found in Wahba (1968). A proof for the other cases may be carried out similarly.

3. DISTRIBUTION THEORY

3.1. Distribution Theory for \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \)

Let

\[
\bar{\lambda}_1 = |\hat{F}(\omega)| \prod_{j=1}^{P} f_j(\omega), \quad l = 1, 2, \ldots, M,
\]

\[
\bar{\lambda}_2 = |\hat{F}(\omega)| / f_{11}(\omega) | \bar{F}_{\#1}(\omega)|.
\]

Suppose \( S = \{s_{ij}\} \) is a \( P \times P \) real Wishart matrix estimated on \( n' \) degrees of freedom, say a sample covariance matrix,

\[ S \sim W(S, P, n'). \]

Let

\[
\begin{pmatrix}
\hat{S}_{11} & \hat{S}_{12} \\
\hat{S}_{12} & \hat{S}_{22}
\end{pmatrix}.
\]

\[ C_{M,n,P} = \Phi_{1}(s) = \phi_{1}(s)
\]

\[ \Phi_{1}(s) \text{ may be inverted by } \frac{P}{3} \text{ the density } f_{1}(x) \text{ for } \]

\[ f_{1}(x) = (n'-1)^{3M(n'-2)} + \sum_{i=1}^{M-1} (n'-i)^{M-i} (2^i). \]

\[ e^{0} \text{, } x < 0. \]
As is well known (see Anderson, 1958), the likelihood-ratio statistics for the hypotheses
\[ A_1: \quad S \text{ diagonal,} \]
\[ A_2: \quad s_{ij} = 0, \quad j = 2, 3, \ldots, P, \quad S = \{s_{ij}\}, \]
are, respectively, monotone functions of
\[ \eta_1 = |S| \prod_{i=1}^P |s_{ii}| \]
\[ \eta_2 = |S| |s_{11}| |S_{22}|; \]
and when \( A_1 \) and \( A_2 \), respectively, are true, it is well known that \( \eta_1 \) and \( \eta_2 \) are distributed as
\[ \eta_1 \sim \prod_{j=1}^{P-1} \beta_{1(n' - j, j)} \]
\[ \eta_2 \sim \beta_{1(n' - (P-1), (P-1)}}, \]
where \( \beta_{\mu, \nu} \) are independent beta random variables with \( \mu \) and \( \nu \) degrees of freedom. By the same techniques used in Anderson (1958) it is straightforward to show that when \( \bar{H}_1, \bar{H}_2 \), respectively, are true,
\[ \lambda_{1l} \sim \prod_{j=1}^{P-1} \beta_{1(n' - j, j)}, \quad l = 1, 2, \ldots, M, \]
(3.3)
\[ \lambda_{1l} \sim \beta_{1(n' - (P-1), (P-1)}}, \quad l = 1, 2, \ldots, M, \]
(3.4)
where we have written \( n' = 2n + 1 \). (The distribution of \( \lambda_{1l} \) under the null and alternative is given by Goodman (1963).) Since the characteristic function \( \Phi_{1}(s) \) of \( -\log \beta_{\mu, \nu} \) is
\[ \Phi_{1}(s) = \frac{\Gamma(\mu + \nu)}{\Gamma(\mu) \Gamma(\mu + \nu - is)}, \]
(3.5)
and \( \lambda_{1l} \sim \beta_{1(n' - (P-1), (P-1)} \), we have (under \( \bar{H}_3 \)) that the characteristic function \( \Phi_{1}(s) \) of \( t_1 = -(1/M) \log \lambda_{1l} \) is given by
\[ \Phi_{1}(s) = C_{M, n', P} \prod_{j=1}^{P-1} \frac{\Gamma(n' - j - (is/M))}{\Gamma(n' - (is/M))} \]
\[ = C_{M, n', P} \left[ \prod_{j=1}^{P-1} (n' - j - (is/M))^{(P-j)M} \right]^{-1}, \]
(3.6)
\[ C_{M, n', P} = \left[ \Gamma(n')^{(P-1)M} / \prod_{j=1}^{P-1} j^{n' - j} \right]^{M}. \]
\( \Phi_{1}(s) \) may be inverted by standard formulae (Bateman, 1954). For example, for \( P = 3 \) the density \( f_1(x) \) for \( t_1 = -(1/M) \log \lambda_{1l} \) is
\[ f_1(x) = (n' - 1)^3 \sum_{l=1}^{3} \frac{(-1)^M x^M}{(M-l)!} \frac{M + l - 2}{M} \exp \left\{ -(n' - 1) \frac{x}{M} \right\} \]
\[ + \sum_{l=1}^{3} \frac{(-1)^M x^M}{(M-l)!} \frac{M + l - 2}{M} \exp \left\{ -(n' - 2) \frac{x}{M} \right\}, \quad x > 0, \]
(3.7)
Similarly, the characteristic function \( \Phi_2(s) \) of \( t_2 = (1/M) \log \bar{\lambda}_2 \) under \( \bar{H}_2 \) is given by

\[
\Phi_2(s) = D_{M,n',P} \left[ \frac{\Gamma(n' - (P-1) - (is/M))}{\Gamma(n' - (is/M))} \right]^M
\]

\[
= D_{M,n',P} \prod_{j=1}^{P-1} (n' - j - (is/M))^{-M},
\]

(3.8)

where

\[
D_{M,n',P} = \left[ \frac{\Gamma(n')}{\Gamma(n' - (P-1))} \right]^M.
\]

The density \( f_2(x) \) for \( t_2 = -(1/M) \log \bar{\lambda}_2 \) may be formed by the same formulae, the result for \( P = 3 \) is

\[
f_2(x) = (n' - 1)^M (n' - 2)^M \sum_{i=1}^{M} \frac{1}{(M-i)!} \left( \frac{M+i-2}{i-1} \right) \left( \frac{x}{M} \right)^{i-1}
\]

\[
\times \left[ (-1)^M \exp \left\{ -M(1 - \frac{x}{M}) \right\} + (-1)^{i-1} \exp \left\{ -(n' - 2) \frac{x}{M} \right\} \right].
\]

(3.9)

Using (3.6) and (3.8) the null means and variances of \( t_1 \) and \( t_2 \) are given by

\[
E t_1 = \sum_{j=1}^{P-1} \frac{1}{(n' - j)},
\]

\[
\text{var} t_1 = \frac{1}{M} \sum_{j=1}^{P-1} \frac{1}{(n' - j)^2} = O \left( \frac{1}{M n'^2} \right),
\]

\[
E t_2 = \sum_{j=1}^{P-1} \frac{1}{(n' - j)},
\]

\[
\text{var} t_2 = \frac{1}{M} \sum_{j=1}^{P-1} \frac{1}{(n' - j)^2} = O \left( \frac{1}{M n'^2} \right).
\]

(3.10)

Choose \( n \) (large) fixed and let \( M \to \infty \), then \( t_1 \) and \( t_2 \) are both asymptotically normal by the central limit theorem. Expressions for the mean and variance of \( t_2 \) under the alternative may be obtained using the density of \( \bar{\lambda}_2 \) given by Goodman. The expressions are unwieldy and will not be reproduced here.

A simpler procedure seems to be to consider \( M \) large fixed, and let \( n' \to \infty \). Then, under the general alternative to \( \bar{H}_1 \), we have, by Theorem 4.2.5 of Anderson (1958), that

\[
\sqrt{n'} \left[ -\frac{1}{M} \log \bar{\lambda}_1 + \frac{1}{M} \sum_{i=1}^{M} \log \left\{ |F(\omega)| \right\} \right]
\]

(3.11)

is asymptotically normal with zero mean. Carrying out the differentiation required by that theorem, it is possible to show that the asymptotic variance is

\[
\frac{1}{M^2} \sum_{i=1}^{M} \sum_{i' < j} \left| W_{ij}(\omega) \right| \approx \frac{1}{2 \pi M} \int_{0}^{2\pi} |\sum_{i=1}^{M} |W_{ij}(\omega)| \, d\omega,
\]

(3.12)

where \( W_{ij}(\omega) = |f_{ij}(\omega)|^2 / f_{k}(\omega) \) for \( M \) time series \( X_1(t) \) and \( X_2(t) \) is given.

Under the general alternative Theorem 4.2.5 of Anderson (1958)

\[
\sqrt{n'} \left[ -\frac{1}{M} \right]
\]

is asymptotically normal for large

\[
\frac{2}{M^3} \sum_{i=1}^{M} W_{i1,2,3,\ldots,i} F(\omega),
\]

where \( W_{i1,2,3,\ldots,i}(\omega) \), the partial c.c., is given by

\[
W_{i1,2,3,\ldots,i}(\omega) = F(\omega),
\]

(3.2)

Consider \( M \) independent

\[
l = 1, 2, \ldots, M. \]

Estimate \( \sigma^2 \) by \( \hat{\sigma}^2 \) of freedom. Then

\[
V = \frac{\hat{\sigma}^2}{\hat{\sigma}^2}
\]

is, except for a constant, the variances. Hence \( \bar{\lambda}_2 \) is distributed

The joint density of \( \bar{f}_{il} \), \( l = 1, 2, \ldots, M \),

\[
p(f_{11}, f_{12}, \ldots)
\]

where \( n' = 2n + 1 \).

Now let \( \bar{c} = \frac{1}{2}[\bar{f}_i - 1] \).

Then, siring the multiplicative factor applied to \( j \) log
where $W_{ij}(\omega) = |f_i(\omega) f_j^*(\omega)|^2 |f_i(\omega)| f_j(\omega)$, and $W_{ij}(\omega)$, the coherence between the two time series $X_i(t)$ and $X_j(t)$, is given by $W_{ij}(\omega) = |f_i(\omega) f_j^*(\omega)|^2 f_j(\omega)$.

Under the general alternative to $H_0$, for $M$ (large) fixed, we have, again by Theorem 4.2.5 of Anderson (1958), as $n' \to \infty$, that
\[ n' \left( -\frac{1}{M} \log \lambda_2 - \sum_{i=1}^M \log |W_{1,2,3,\ldots,t}(\omega)| \right) \]

is asymptotically normal for large $n'$, with 0 mean, where $W_{1,2,3,\ldots,t}(\omega)$ is given by
\[ W_{1,2,3,\ldots,t}(\omega) = |F(\omega)| f_1(\omega) F_2(\omega). \]

The asymptotic variance can be shown to be
\[ \frac{2}{M^2} \sum_{i=1}^M W_{1,2,3,\ldots,t}(\omega) \approx \frac{1}{2\pi M} \int_0^{2\pi} W_{1,2,3,\ldots,t}(\omega) d\omega, \]
where $W_{1,2,3,\ldots,t}(\omega)$, the partial coherence between $X_1(t)$ and $X_j(t)$, $j = 2, 3, \ldots, P$, is given by
\[ W_{1,2,3,\ldots,t}(\omega) = |F(\omega)| f_1(\omega) F_2(\omega). \]

3.2. Distribution Theory for $\lambda_2$

Consider $M$ independent populations, the $l$th population being $N(\mu_l, \sigma^2_l)$, $l = 1, 2, \ldots, M$. Estimate $\sigma^2_l$ by $\bar{S}_l^2$, the sample variance, estimated with $2(2n+1)$ degrees of freedom. Then $V$, where
\[ V = \left[ \prod_{l=1}^M \frac{\bar{S}_l^2}{\bar{S}_l^2} \right]^{2n+1}, \]
is, except for a constant, the well-known Bartlett statistic for homogeneity of variances. Hence $\lambda_2$ is distributed as $M^M V^{(2n+1)}$. For notational convenience, let
\[ f_l(\omega) = f_l, \]
\[ f_l^*(\omega) = f_l. \]

The joint density of $f_l$, $l = 1, 2, \ldots, M$, is then
\[ p(f_1, f_2, \ldots, f_M) = \prod_{l=1}^M \left( \frac{1}{\pi \sigma_l^2} \right)^{n_l/2} \exp \left( -\frac{f_l^2}{\sigma_l^2} \right), \]
where $n_l = 2n+1$.

Now let $\xi_l = 1/f_l$, $\bar{c} = 1/\left[ \max_l \xi_l + \min_l \xi_l \right]$ and let $\xi_l = \bar{c} \xi_l$, $b_l = (\bar{c} - \xi_l)/\bar{c}$, where $|b_l| < 1$, $l = 1, 2, \ldots, M$. Then, since the distribution of $\lambda_2$ is invariant under a constant multiplicative factor applied to $f_l$, $\log \lambda_2$ is distributed as
\[ \log \left( \prod_{l=1}^M \xi_l \right) \left( \frac{1}{M} \sum \xi_l \right)^M, \]
where the joint density of \( \{ \xi_i \}_{i=1}^{M} \) is given by

\[
p(\xi_1, \xi_2, \ldots, \xi_M) = \frac{1}{(\Gamma(n'))^M} \prod_{k=1}^{M} (1 - b_v) n' \prod_{i=1}^{M} \xi_i^{n'-1} \exp \left( - \sum_{i=1}^{M} \xi_i + \sum_{v=1}^{M} b_v \xi_v \right) \tag{3.19}\]

In the appendix it is shown that the characteristic function \( \Phi_0(s) \) of \( \log \xi_0 \) is given by

\[
\Phi_0(s) = E \exp(is \log \xi_0) = \Phi_0(s) g(s),
\]

where

\[
\Phi_0(s) = M^{isM} \frac{\Gamma(Mn') \Gamma(is + n')^M}{\Gamma(n')^{3M} \Gamma((is + n')^M)} \tag{3.20a}
\]

is the characteristic function of \( \log \xi_0 \) under the null hypothesis and

\[
g(s) = \prod_{p=1}^{M} (1 - b_v) n' \sum_{k=0}^{\infty} \psi_k(s) \theta_k(s), \tag{3.20b}
\]

where

\[
\psi_k(s) = \frac{\Gamma((is + n')^M)}{\Gamma((is + n')^{M+k})} \frac{\Gamma(Mn' + k)}{\Gamma(Mn')} \tag{3.20c}
\]

and \( \theta_k(s) \) is defined by

\[
\sum_{k=0}^{\infty} \theta_k(s) t^k = \prod_{p=1}^{M} (1 - b_v) t^{-i(is + n')}, \quad |t| < 1. \tag{3.20d}
\]

An expression for \( \Phi_0(s) \) for \( n = 1 \) was given by Whittle in 1950, when he suggested the use of \( \tilde{\lambda}_0 \) (with \( n = 1 \)) as a statistic for testing the independence of residuals after model fitting. Using the duplication formula for the gamma function (Whittaker and Watson, 1927),

\[
\Gamma(Mz) = \{M^{-3/2}(2\pi)^{1/2(M-1)}\}^{-1} \Gamma(z) \Gamma\left( z + \frac{1}{M} \right) \cdots \Gamma\left( z + \frac{M-1}{M} \right),
\]

it follows from (3.5) and (3.20a) that \( \tilde{\lambda}_0 \) under the null hypothesis, is distributed as

\[
\tilde{\lambda}_0 \sim \prod_{i=1}^{M} \beta_{n',0,i}^{0}, \tag{3.21}
\]

where \( \{ \beta_{n',0,i,0}^{0} \}_{i=1}^{M} \) are \( M \) independent \( \beta_{n',0,i,0}^{0} \) random variables.

The moments of \( \log \tilde{\lambda}_0 \) under the null hypothesis, are readily obtainable by using the formula for the logarithmic derivative of the gamma function (Whittaker and Watson, 1927), which gives

\[
\frac{\partial}{\partial s} \left|_{s=0} \right. \log \left[ \frac{\Gamma((is + n')^M)}{\Gamma((is + n')^{M+j})} \right] = \left( -iM \right)^{(l-1)} \left( \frac{1}{M-1} \sum_{j=0}^{n'+j} - \sum_{j=0}^{Mn'+j} \right). \tag{3.22}
\]

Although Bartlett's statistic has been with us since 1937, its distribution theory does not seem to be in a completely satisfactory state. Let \( W_1 = \tilde{\lambda}_0^{0} \). An asymptotic expansion, for the c.d.f. of \( W_1 \), for large \( n' \), in terms of \( \chi^2 \) c.d.f.'s is given in Anderson (1958, p. 225, equation (7)), which should suffice for practical purposes.
1971]

WAHBA – Stationary Multivariate Time Series 163

For sufficiently small \( b_n \), \( \log g(s) \) can be expanded up to the second order in \( \{ b_n \} \) as

\[
\log g(s) = \log \prod \left( 1 - b_n s \right)^2 + \log \left( 1 + \sum_{q=1}^{\infty} \left( \sum_{r=1}^{M} \xi_r \theta^q \right)^2 \right)
\]

\[
\approx \text{const} + \sum_{q=1}^{\infty} \left( \sum_{r=1}^{M} \xi_r \theta^q \right)^2 + \ldots
\]

\[
= \text{const} + \sum_{q=1}^{\infty} \left( \sum_{r=1}^{M} \xi_r \theta^q \right)^2 + \ldots,
\]

(3.23)

where \( b = (1/M) \sum b_n \). Hence, for \( \{ b_n \} \) sufficiently small,

\[
\begin{align*}
E \log \lambda_0 & \approx \mu_0 - \frac{M^2 n}{2(Mn + 1)} \left( \sum_{n=1}^{\infty} \frac{1}{M} \sum_{r=1}^{M} (b_n - b)^2 \right), \\
\var \log \lambda_0 & \approx \sigma_0^2 + \frac{M^2 n}{(Mn + 1)^2} \left( \sum_{n=1}^{\infty} \frac{1}{M} \sum_{r=1}^{M} (b_n - b)^2 \right) = O(M/n),
\end{align*}
\]

(3.24)

where \( \mu_0 \) and \( \sigma_0^2 \) the null mean and variance, are, from (3.20a) and (3.22),

\[
\begin{align*}
\mu_0 &= M \left( \log M - \sum_{j=0}^{n-1} \frac{1}{j} \right), \\
\sigma_0^2 &= M \sum_{j=0}^{\infty} \frac{1}{(n+j)^2} - M^2 \sum_{j=0}^{\infty} \frac{1}{(Mn+j)^2}.
\end{align*}
\]

(3.25)

It is possible to show that the neglected terms in (3.24) are \( O(M/n) \) times higher order terms in \( \{ b_n \} \); the details are omitted.

For sufficiently small \( \{ b_n \} \),

\[
\frac{1}{M} \sum_{n=1}^{\infty} \frac{1}{(Mn+j)^2} \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} ||f||_2 (\omega - f)^2 (f)^2 d\omega.
\]

(3.26)

A detailed examination of the power of this test against nearby alternatives might begin with equation (A.12), where \( \Phi_k(s) \) is expressed as a weighted sum of characteristic functions, the weights being a constant multiple of terms in the power series expansion of \( \prod_{n=1}^{M} (1-b_n s)^n \). This investigation is omitted.

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**APPENDIX**

*The Characteristic Function of \( \log \tilde{X}_n \)*

Let \( X_n, \nu = 1, 2, \ldots, M, \) be independent with the density \( f_n(x) \) of \( X_n \) given by

\[
f_n(x) = \frac{(1-b_n)^n}{\Gamma(n)} x^{n-1} \exp\{- (1-b_n)x \}, \quad x > 0, \quad |b_n| < 1, \quad \nu = 1, 2, \ldots, M,
\]

and let

\[
\tilde{X}_n = \prod_{\nu=1}^M X_n / \left( \frac{1}{M} \sum_{\nu=1}^M X_n \right)^M,
\]

\[
\Phi_n(s) = E \exp(\nu \log \tilde{X}_n) = E(\tilde{X}_n)^{is} = M^{isM} \prod_{\nu=1}^M (1-b_n)^n \Gamma(n)^{-M} \int_0^{\infty} \cdots \int_0^{\infty} \left( \prod_{\nu=1}^M X_n \right)^{is+n-1} \left( \sum_{\nu} X_n \right)^{-isM} \times \exp\left(-\sum_{\nu} X_n + \sum_{\nu} b_n X_n \right) \prod_{\nu} dx_n, \quad (A.1)
\]

Making the change of variables \( Y_n = \sum_{\nu=1}^M X_n, \nu = 1, 2, \ldots, M-1, Z_M = Y_M \) gives

\[
X_n = (1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \quad Z_0 = 0, \quad \nu = 1, 2, \ldots, M,
\]

\[
\prod_{\nu=1}^M dX_n = \prod_{\nu=1}^M Z_{n-1} dZ_n
\]

and

\[
\Phi_n(s) = M^{isM} \prod_{\nu=1}^M (1-b_n)^n \left( \Gamma(n) \right)^{-M} \left[ \int_0^{\infty} Z_M^{M-1} \exp(-Z_M) dZ_M \right] \times \prod_{\nu=1}^{M-1} \int_0^{\infty} dZ_n Z_n^{(is+n-1)}(1-Z_n)^{is+n-1} \times \exp\left[ \sum_{\nu=1}^M b_n(1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \right]. \quad (A.2)
\]

Expand the second exponential in a power series, the \( k \)th term of which is

\[
\frac{1}{k!} \left\{ \sum_{\nu=1}^M b_n(1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \right\}^k.
\]

The coefficient of \( b_1 b_2 \ldots b_k q_1 q_2 \ldots q_k \)

\[
q_1 q_2 \ldots q_k = 0, \quad \nu = 1, 2, \ldots, \infty,
\]

and the coefficient of \( b_1^2 b_2 \ldots b_k q_1 q_2 \ldots q_k \) in (A.2) is

\[
\left( \prod_{\nu=1}^M q_n! \right) \int_0^{\infty} dZ_M
\]

\[
+ (1-z) \Gamma(Mn+k)
\]

\[
\times \Gamma(is+n)
\]

\[
= \left( \Gamma(Mn+k) \right) \Phi_n(s), \quad (A.4)
\]

Rewriting (A.4) with the use of

\[
\sum_{\nu=1}^M q_n = k
\]

the characteristic function

\[
\Phi_n(s) = \Psi_n(s) \quad \Phi_n(s) = \Psi_n(s) \prod_{\nu=1}^M (1-b_n)^{-n} \Gamma(n)^{-M} \left[ \int_0^{\infty} Z_M^{M-1} \exp(-Z_M) dZ_M \right] \times \prod_{\nu=1}^{M-1} \int_0^{\infty} dZ_n Z_n^{(is+n-1)}(1-Z_n)^{is+n-1} \times \exp\left[ \sum_{\nu=1}^M b_n(1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \right]. \quad \Phi_n(s) = \Psi_n(s) \quad \Phi_n(s) = \Psi_n(s) \prod_{\nu=1}^M (1-b_n)^{-n} \Gamma(n)^{-M} \left[ \int_0^{\infty} Z_M^{M-1} \exp(-Z_M) dZ_M \right] \times \prod_{\nu=1}^{M-1} \int_0^{\infty} dZ_n Z_n^{(is+n-1)}(1-Z_n)^{is+n-1} \times \exp\left[ \sum_{\nu=1}^M b_n(1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \right]. \quad (A.2)
\]

where

\[
\Psi_n(s) = \Psi_n(s)
\]

Now, for \( |b_n| < 1, \nu = 1, 2, \ldots \),

\[
\prod_{\nu=1}^M (1-b_n)^{-n} \Gamma(n)^{-M} \left[ \int_0^{\infty} Z_M^{M-1} \exp(-Z_M) dZ_M \right] \times \prod_{\nu=1}^{M-1} \int_0^{\infty} dZ_n Z_n^{(is+n-1)}(1-Z_n)^{is+n-1} \times \exp\left[ \sum_{\nu=1}^M b_n(1-Z_{n-1}) \prod_{\nu=1}^M Z_{n-1} \right]. \quad (A.2)
\]

Hence, we may write

\[
\Phi_n(s) = \Psi_n(s)
\]

where \( \theta_k(s) \) is defined by

\[
\sum_{k=1}^{\infty} \theta_k(s)
\]
The coefficient of \( b_1 b_2 \ldots b_M \), \( \sum_{n=1}^{M} q_n = k \) in the expansion of this term is given by

\[
\frac{1}{q_1 \cdots q_M} \prod_{n=1}^{M} Z_n^{1+q_n} \prod_{n=1}^{M-1} (1-Z_n)^{q_{n+1}},
\]

(A.3)

and the coefficient of \( b_1 b_2 \ldots b_M \), \( \sum_{n=1}^{M} q_n = k \) in the expansion of the term in brackets in (A.2) is

\[
\left( \frac{1}{\prod_{n=1}^{M} q_n} \right) \int_{0}^{\infty} dZ_M \exp \left( -Z_M \sum_{n=1}^{M-1} \prod_{n=1}^{M-1} Z_n^{1+q_n} \prod_{n=1}^{M-1} (1-Z_n)^{q_{n+1}} \right)
\]

\[
+ \prod_{n=1}^{M} (1-Z_n)^{q_{n+1}}, dZ_M
\]

\[
= \left[ \Gamma(Mn+k) \prod_{n=1}^{M} q_n \prod_{n=1}^{M} \Gamma \left( (is+n) + \sum_{j=1}^{n} q_j \right) \right]
\]

\[
\times \prod_{n=1}^{M} \Gamma((is+n+q_{n+1})/\Gamma((is+n) + \sum_{j=1}^{n} q_j))
\]

\[
= \left[ \Gamma(Mn+k) \prod_{n=1}^{M} q_n \prod_{n=1}^{M} \Gamma((is+n+q_{n+1})/\Gamma((is+n) + k)) \right].
\]

(A.4)

\( \Phi_\theta(s) \), the characteristic function of \( \log \lambda_0 \) under \( \mathcal{H}_0 \) is, from (A.2) with \( |b_1| = 0 \),

\[
\Phi_\theta(s) = M^{isaM} \left( (\Gamma(is+n)M)^{M} \Gamma(Mn) \right) \prod_{n=1}^{M} \Gamma((is+n) + k) \prod_{n=1}^{M} \Gamma((is+n) + M + k).
\]

(A.5)

Rewriting (A.4) with the use of (A.5), the coefficient of \( b_1 b_2 \ldots b_M \) in \( \Phi_\theta(s) \), \( \sum_{n=1}^{M} q_n = k \) may be written

\[
\Phi_\theta(s) \prod_{n=1}^{M} (1-b_n)^{n} \Psi/(s) \prod_{n=1}^{M} \left( (is+n-1+q_n) \right),
\]

(A.6)

where

\[
\Psi/(s) = \frac{\Gamma((is+n)M)}{\Gamma((is+n) + k)}
\]

(A.7)

Now, for \( |b_n| < 1, n = 1, 2, \ldots, M, |t| < 1 \), we have

\[
\prod_{n=1}^{M} (1-b_n)^{n} = \sum_{k=0}^{\infty} \sum_{n=1}^{M} \left( (is+n-1+q_n) \right) b_n^k,
\]

(A.8)

where \( \sum_{q_n} \) is the sum over all partitions \( (q_1, q_2, \ldots, q_M) \) of \( k \) such that \( \sum_{n=1}^{M} q_n = k \). Hence, we may write

\[
\Phi_\theta(s) = \Phi_0(s) \prod_{n=1}^{M} (1-b_n)^{n} \sum_{k=0}^{\infty} \Psi/(s) \theta_k(s),
\]

(A.9)

where \( \theta_k(s) \) is defined by

\[
\sum_{k=1}^{\infty} \theta_k(s) t^k = \prod_{n=1}^{M} (1-b_n)^{n} \prod_{n=1}^{M} (1-t)^{n}, \quad |t| < 1.
\]

(A.10)
\( \Phi_3(s) \) may also be expressed as a weighted average of characteristic functions. Let

\[
\Phi_{q_1,q_2,\ldots,q_M}(s) = \left( \prod_{r=1}^{M} \Gamma(n+q_r) \right) \left[ \prod_{r=1}^{M} \Gamma((is+n+q_r)/M) M+k \right] \\
= \prod_{\nu=1}^{M-1} \exp \{ is \log Z_\nu(1-Z_\nu) \},
\]

where \( Z_\nu \) is distributed as \( \beta_{\nu \nu} \) with \( \xi = \nu n + \sum_{j=1}^{\nu} q_j \), \( \eta = n+q_{\nu+1} \). Then

\[
\Phi_2(s) = M^{isM} \prod_{\mu=1}^{M} (1-b_{\mu})^n \sum_{k=0}^{\infty} \sum_{\nu_k} \Phi_{q_1,q_2,\ldots,q_M}(s) \prod_{r=1}^{M} \left( n-1+q_r \right) b_{\nu_k}.
\]

The author is grateful to Dr I for the following error in the proof that \( n^{-3} \sum E[Z_i] (i = 1, \ldots, n) \) not ruled out by Assumption 1.3 should the Assumption 1.3 should the

Assumption 1.3. Assume that for all \( \delta \) in an open interval \( \mu(\delta, \tau) = E_{\theta_{\nu}}, \)

and

\( \sigma^2(\delta, \tau) = E_{\theta_{\nu}} \)

are continuous functions of \( \tau \).

This reformulation of Ass of Section 4.