REGRESSION DESIGN FOR SOME EQUIVALENCE CLASSES OF KERNELS

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Earlier results on asymptotically optimal sequences of regression designs for autoregressive stochastic processes are extended (nearly) to the equivalence classes of such processes.

1. Introduction. Let \( \{Y(t), \ t \in T\} \) be a stochastic process of the form

\[
Y(t) = \theta f(t) + X(t)
\]

where \( \theta \) is an unknown constant, \( f(t) \) is a known function on \( T \), \( T \) is a closed bounded interval which we take to be \([0, 1]\), and \( \{X(t), \ t \in T\} \) is a zero-mean Gaussian stochastic process with known continuous covariance kernel \( Q \), \( EX(t)X(t') = Q(t, t') \). The regression design problem is to choose an \( n \)-point subset (or “design”) \( T_n \),

\[
T_n = \{t_1 < t_2 < \cdots < t_n, \ t_i \in T\}
\]

so that the variance \( \sigma^2_{T_n} \) of the Gauss–Markov estimate of \( \theta \) given \( \{Y(t), \ t \in T_n\} \) is as small as possible.

This problem has been considered by Sacks and Ylvisaker, Wahba, and Hájek and Kimeldorf [3], [11], [12], [13], [14], [16] for various special cases of \( Q \). It is known that \( \sigma^2_{T_n} \) is bounded away from 0 as \( \Delta = \max_i |t_{i+1} - t_i| \) tends to 0 if and only if \( f \in Q \), where \( H_Q \) is the unique reproducing kernel Hilbert space (RKHS) with reproducing kernel (RK)\( Q \), see [8]. It will be assumed that the reader is familiar with the basic properties of RKHS as given in [8], [16], see also [1].

For fixed \( t \in T \), let \( \langle \cdot, \cdot \rangle_{Q} \) represent the evaluation functional at \( t \) in \( H_Q \), that is

\[
\langle Q_t, f \rangle_{Q} = f(t), \quad f \in H_Q
\]

and

\[
Q_t(t') = Q(t, t')
\]

where \( \langle \cdot, \cdot \rangle_{Q} \) is the inner product in \( H_Q \).

Let \( P_{T_n} \) be the projection operator in \( H_Q \) onto the subspace spanned by \( \{Q_t, \ t \in T_n\} \). It is well known that if \( f \in H_Q \), then \( \sigma^2_{T_n} = \|P_{T_n}f\|_Q^2 \) and \( \sigma^{-2} = \|f\|_Q^2 \), where \( \|\cdot\|_Q \) is the norm in \( H_Q \) and \( \sigma^2 \) is the variance of the Gauss Markov estimate of \( \theta \), given \( \{Y(t), \ t \in T\} \). Hence \( \sigma^2_{T_n} \) is minimized by minimizing \( \|f - P_{T_n}f\|_Q^2 \). From this point of view, the problem becomes one of choosing

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an optimal subspace in $H_Q$ of the form span \{Q_t, t \in T_n\}, for the purpose of approximating the given element $f$. In this context, the problem has been considered by Karlin [5], [6]. The solution also has applications to the approximate solution of linear differential and integral equations, see [17], [18].

We suppose that $\{X(t), t \in T\}$ has exactly $m - 1$ quadratic mean derivatives. This entails that the functions $Q_t^{(v)}(\cdot)$ defined by
\[
Q_t^{(v)}(\cdot) = \left(\frac{\partial^v}{\partial s^v}\right)Q(s, \cdot)\bigg|_{s=t}
\]
are all well defined and in $H_Q$, for $t \in T$ and $v = 1, 2, \ldots, m - 1$. Let $P_{m,T_n}$ be the projection operator in $H_Q$ onto the subspace of $H_Q$ spanned by
\[
\{Q_t^{(v)}, t \in T_n, v = 0, 1, \ldots, m - 1\}.
\]
The optimal experimental design problem becomes tractable if we attempt to minimize $\|f - P_{m,T_n}f\|_Q$ rather than $\|f - P_{T_n}f\|_Q$, and the results are still useful, because of the relation ([16], (1.15))
\[
\inf_{T_n} \|f - P_{m,T_n}f\|_Q \leq \inf_{T_n} \|f - P_{m,T_n}f\|_Q \leq \inf_{T_n} \|f - P_{T_n}f\|_Q.
\]

Further information about the role of derivatives may be found in Karlin [6], especially Theorem 3(i) and Theorem 4, and Sacks and Ylvisaker [13]. In particular, ([6], equation (13), [13], Theorem 4) if $m = 2$ and other conditions are satisfied, the right hand inequality in (1.2) becomes an equality.

Following [13], a sequence $T_n^*, n = 1, 2, \ldots$ of designs is said to be asymptotically optimal (with derivatives) if
\[
\lim_{n \to \infty} \frac{\|f - P_{m,T_n^*}f\|_Q}{\inf_{T_n} \|f - P_{m,T_n}f\|_Q} = 1.
\]

In [16], asymptotically optimal designs (with derivatives) are found for the case where $X$ is a stochastic process formally satisfying the stochastic differential equation
\[
\begin{align*}
(L_m X)(t) &= dW(t), \quad t \in [0, 1] \\
X^{(v)}(0) &= \xi_v, \quad \nu = 0, 1, \ldots, m - 1,
\end{align*}
\]
where $L_m$ is defined by
\[
(L_m f)(t) = \sum_{j=0}^m a_{m-j} f^{(j)}(t),
\]
$\{W(t), t \in [0, 1]\}$ is a Wiener process and $\{\xi_v\}_{v=0}^m$ are $m$ zero mean Gaussian random variables independent of $W(t), t \in [0, 1]$. $L_m$ (in [16]) is such that its null space is spanned by $\{\phi_v\}_{v=0}^m$ where $\{\phi_v\}_{v=1}^m$ is an extended, complete Tchebychev (ECT) system of continuity class $C^m$. In [3], the conditions on $L_m$ are relaxed to: $a_0 \neq 0, a_{m-j} \in C^j$, with $E\xi_v^2 = 0$. It is the purpose of this note to show that the results of [3] and [16] may be extended to “nearly all” stochastic processes equivalent to $X$ of (1.3).
A sequence of designs may be conveniently described by a continuous positive density $h$ on $T = [0, 1]$. Let $T_n = T_n(h) = \{t_{0n}, t_{1n}, \ldots, t_{nn}\}$ be defined by

$$\int_{t_{im}}^t h(x) \, dx = \frac{i}{n},$$

$i = 0, 1, \ldots, n$.

(For ease of notation we are now letting $T_n$ contain $n + 1$ points.)

We have the following result from [16] (as a consequence of Lemma 3).

**Proposition 1.** Let $X$ be as in (1.3), and suppose

$$f(t) = \int_0^t Q(t, s)\rho(s) \, ds$$

where $\rho > 0$, and $\rho$ possess a bounded first derivative. Let $T_n = T_n(h)$. Then

$$||f - P_{m, T_n}f||_Q^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(2m)! (2m + 1)!} \int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)} \, ds + o\left(\frac{1}{n^{2m}}\right)$$

where

$$\alpha(s) = \frac{1}{a_0^2(s)}.$$

Following [11], asymptotically optimal sequences of designs are found from (1.4) by using a Hölder inequality and the fact that $\int_0^1 h(s) \, ds = 1$ to show that

$$\int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)} \, ds \geq \left[\int_0^1 \left[\rho^2(s)\alpha(s)\right]^{1/(2m+1)} \, ds\right]^{2m+1}$$

with equality iff

$$h(s) = \frac{\left[\rho^2(s)\alpha(s)\right]^{1/(2m+1)}}{\int_0^1 \left[\rho^2(u)\alpha(u)\right]^{1/(2m+1)} \, du}.$$

Thus if

$$\int_0^1 \left[\rho^2(s)\alpha(s)\right]^{1/(2m+1)} \, ds = \frac{i}{n} \int_0^1 \left[\rho^2(s)\alpha(s)\right]^{1/(2m+1)} \, ds,$$

$i = 0, 1, \ldots, n$ then $T_n^* = \{t_{0n}^*, t_{1n}^*, \ldots, t_{nn}^*\}, n = 1, 2, \ldots$, is an asymptotically optimal sequence of designs with

$$||f - P_{m, T_n^*}f||_Q^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(2m)! (2m + 1)!} \left[\int_0^1 \left[\rho^2(s)\alpha(s)\right]^{1/(2m+1)} \, ds\right]^{2m+1} + o\left(\frac{1}{n^{2m}}\right).$$

The "parameter function" $\alpha(s), s \in [0, 1], \alpha(s) = 1/a_0^2(s)$, plays a central role in the solution. It is not hard to convince one's self (see e.g. Hájek [2]) that two stochastic processes of the form (1.3) considered in [16] are equivalent iff their "initial value" rv's (1.3 b) are equivalent and the leading coefficient $a_0(s)$ of the defining differential operator is the same for both processes.

Thus, a maximal generalization of Theorem 1 would appear to be to $X$'s equivalent to those of the form (1.3). In the remainder of this note we show that this is, in fact, the case, modulo a regularity condition on $Q$ which we cannot seem to get rid of.

2. **Equivalence classes of kernels for $X$ of (1.3).** Let $\{X_i(t), t \in [0, 1]\}, i = 0, 1$ be two zero mean Gaussian stochastic processes with continuous covariances
$Q_0(s, t)$ and $Q_1(s, t)$ respectively. Now, let $Q_0$ and $Q_1$ also denote the Hilbert–
Schmidt operators on $L_2[0, 1]$, with Hilbert–Schmidt kernels $Q_0(s, t)$ and $Q_1(s, t)$, 
defined by

$$(Q_i p)(t) = \int_0^1 Q_i(t, s)p(s) ds, \quad p \in L_2[0, 1], \ i = 0, 1.$$ 

A version of the Hájek–Feldman Theorem stated in Root [10] says that the 
measures corresponding to $X_1$ and $X_2$ are equivalent iff

$$(2.1) \quad Q_0^{-1}Q_1Q_0^{-1} = I - B$$

where $Q_i^{-1}$ is the symmetric square root of $Q_i^{-1}$, $i = 0, 1$, and $B$ is a Hilbert–
Schmidt operator with $I - B$ invertible. For simplicity we will say that $Q_0$ and 
$Q_1$ are equivalent if (2.1) holds.

Let

$$(2.2) \quad G_0(s, u) = \left(\frac{s - u}{m - 1}\right)^{m-1} c(u)$$

where

$$c(u) = \frac{1}{a_0(u)}$$

and $(x)_+ = x, x \geq 0$, $(x)_+ = 0$ otherwise. Let

$$(2.3) \quad Q_0(s, t) = \int_0^1 G_0(s - u)G_0(t - u) du .$$

$Q_0$ is the covariance of $X$ of (1.3) with $L_m = a_0D_m$ and $E_{\nu}^{2, \nu} = 0, \nu = 0, 1, \ldots, m - 1$. The Hilbert–Schmidt operator $Q_0$ may be written

$$Q_0 = G_0G_0^*$$

where $G_0^*$ is the adjoint operator to $G_0$, the Hilbert–Schmidt operator with kernel (2.2). Since

$$Q_0 = Q_0^1Q_0^1 = G_0G_0^*,$$

$Q_0^{-1}Q_1Q_0^{-1}$ is unitarily equivalent to $G_0^{-1}Q_1G_0^{*-1}$ and

$$Q_0^{-1}Q_1Q_0^{-1} = I - B$$

with $B$ Hilbert–Schmidt and $I - B$ invertible iff

$$G_0^{-1}Q_1G_0^{*-1} = I + A$$

for $A$ some Hilbert–Schmidt operator with $I + A$ invertible. Thus $Q_0$ and $Q_1$ 
are equivalent if and only if

$$Q_1 = G_0(I + A)G_0^*$$

where $A$ is Hilbert–Schmidt and $I + A$ invertible. We summarize these remarks as

**Proposition 2.** A kernel $Q_1$ is equivalent to $Q_0$ of (2.3) iff

$$(2.4) \quad Q_1(s, t) = \int_0^1 \left(\frac{s - u}{m - 1}\right)^{m-1}\left(\frac{t - u}{m - 1}\right)^{m-1} c(u) du$$

$$+ \int_0^1 \int_0^u \left(\frac{s - u}{m - 1}\right)^{m-1}\left(\frac{t - v}{m - 1}\right)^{m-1} c(u) A(u, v)c(v) du dv$$

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where
\[ \int_0^1 \int_0^1 A(s, t) \, ds \, dt < \infty \]
and \( I + A \) is invertible, \( A \) being the operator with (symmetric) Hilbert–Schmidt kernel \( A(s, t) \).

Now let \( \tilde{Q}_0 \) be
\[ \tilde{Q}_0(s, t) = \sum_{j=0}^{m-1} \phi_j(s)\phi_j(t) + \int_0^s G_0(s, u)G_0(t, u) \, du \]
with \( G_0 \) given by (2.2) and \( \phi_j(s) = s^{j-1}(j - 1)!, j = 1, \ldots, m \). A process \( \{X_0(t), t \in [0, 1]\} \) with covariance (2.5) has a representation
\[ X_0(t) = \sum_{j=0}^{m-1} X_0^{(j)}(0)\phi_j(t) + (X_0(t) - P_{m,0}X_0(t)), \quad t \in [0, 1], \]
where \( P_{m,0}X_0(t) = E\{X_0(t) | X_1^{(j)}(0), j = 0, \ldots, m - 1\} \) and \( \{X_1^{(j)}(0)\}_{j=0}^{m-1} \) are i.i.d. \( \mathcal{N}(0, 1) \). The process \( (X_0(t) - P_{m,0}X_0(t)) \) has covariance \( Q_0 \) of (2.4). For \( \tilde{Q}_1 \) to be equivalent to \( \tilde{Q}_0 \) it is necessary and sufficient that \( \{X_1^{(j)}(0)\}_{j=0}^{m-1} \) exist in q.m. and have a covariance matrix of full rank, and that the process \( X_1(t) - P_{m,0}X_1(t) \) have a covariance \( Q_1 \) of the form (2.4). In this case \( X_1(t) \) has a representation of the form
\[ X_1(t) = \sum_{j=0}^{m-1} X_1^{(j)}(0)\phi_j(t) + (X_1(t) - P_{m,0}X_1(t)) \]
where
\[ \phi_j(t) = \sum_{j=0}^{m-1} \sigma^{j,j} \eta_j(t) \]
with
\[ \{\sigma^{j,j}\} = \{\sigma_{\nu,j}\}^{-1}, \quad \sigma_{\nu,j} = EX_1^{(\nu)}(0)X_1^{(j)}(0), \quad \nu, j = 0, 1, \ldots, m - 1 \]
and
\[ \eta_j(t) = EX(t)X_0^{(j)}(0) = \frac{\partial^{\nu}}{\partial s^\nu} \tilde{Q}_1(t, s) \bigg|_{s=0}, \quad \nu = 0, 1, \ldots, m - 1. \]

By the properties of RKHS (see [8]), the \( \{\eta_j\} \) must all be in \( \mathcal{H}_{\tilde{Q}_1} \), and if \( \tilde{Q}_1 \) is equivalent to \( \tilde{Q}_0 \), they must also be in \( \mathcal{H}_{\tilde{Q}_0} \).

We summarize these remarks in the following

**Proposition 3.** \( \tilde{Q}_1 \) is equivalent to \( \tilde{Q}_0 \) of (2.5) iff
\[ \tilde{Q}_1(s, t) = \sum_{j=0}^{m-1} \tilde{\phi}_j(s)\tilde{\phi}_j(t) + Q_1(s, t), \]
where \( \tilde{\phi}_j^{(\nu)} \) abs. cont., \( \nu = 0, 1, \ldots, m - 1 \), \( \tilde{\phi}_j^{(m)} \in \mathcal{L}_2 \), the \( m \times m \) matrix with \( ij \)th entry \( \sigma_{\nu,j} \),
\[ \sigma_{\nu,j} = \frac{\partial^{\nu+j}}{\partial s^\nu \partial t^j} \left( \sum_{j=0}^{m-1} \tilde{\phi}_j(s)\tilde{\phi}_j(t) \right) \bigg|_{s=0}, \quad \nu, j = 0, 1, \ldots, m - 1 \]
is of full rank, and \( Q_1(s, t) \) is of the form (2.4).

Proposition 3 is a slight generalization of [15], Theorem 8; see also [4].

We have that \( \mathcal{H}_{\tilde{Q}_1} = \mathcal{H}_{\tilde{Q}_0} + \text{span} \{\tilde{\phi}_j\}_{j=1}^{m-1} \), and if \( T_n \) includes the point \( t = 0 \), then
\[ ||f - P_{m,T_n}f||_{\tilde{Q}_1}^2 = ||P_{Q_1}(f - P_{m,T_n}f)||_{\tilde{Q}_1}^2, \]
where \( P_{Q_1} \) is the projection operator.
in $\mathcal{H}_{Q_1}$ onto the subspace $\mathcal{H}_{Q_1}$. Thus we may without loss of generality consider $Q_1$ of the form (2.4). This remark holds, of course, whatever the rank of the matrix $\{\sigma_{ij}\}$.

3. Asymptotically optimal designs for $\tilde{Q}_1$. The purpose of this section is to prove the following

**Theorem.** Let $\tilde{Q}_1$ have a representation

$$\tilde{Q}_1(s, t) = \sum_{i=0}^{m-1} \tilde{\phi}_i(s) \tilde{\phi}_i(t)$$

$$+ \int_0^1 \frac{(s-u)^{m-1}}{(m-1)!} \frac{(t-u)^{m-1}}{(m-1)!} c^2(u) \, du$$

$$+ \int_0^1 \int_0^1 \frac{(s-u)^{m-1}}{(m-1)!} \frac{(t-u)^{m-1}}{(m-1)!} c(u)A(u, v)c(v) \, du \, dv$$

where

(i) $\tilde{\phi}_i^{(s)}$, abs. cont., $\nu = 0, 1, \ldots, m - 1, \tilde{\phi}_i^{(m)} \in L_2[0, 1]$,

(ii) $c > 0$, $c'$ bounded,

(iii) $\int_0^1 \int_0^1 A^2(u, v) \, du \, dv < \infty$,

(iv) the function $\gamma_t$ given by

$$\gamma_t(s) = \int_0^s \frac{\partial}{\partial t} \frac{1}{c(t)} A(t, \eta) c(\eta) \, d\eta$$

is well defined and is in the RKHS $\mathcal{H}_{K_1}$ with RK $K_1$ given by

$$K_1(s, t) = \int_0^{\min(s, t)} c^2(u) \, du + \int_0^1 \int_0^1 c(u)A(u, v)c(v) \, du \, dv$$

and

$$\|\gamma_t\|_{K_1} \leq M_1 < \infty$$

where $\|\cdot\|_{K_1}$ is the norm in $\mathcal{H}_{K_1}$.

Let

$$f(t) = \int_0^t \tilde{Q}_1(t, s) \rho(s) \, ds$$

with $\rho > 0$, $\rho'$ bounded, and let $T_n = \{t_i\}_{i=0}^n$ with

$$\int_0^1 h(u) \, du = \frac{i}{n}, \quad i = 0, 1, \ldots, n$$

where

$$\int_0^1 h(u) \, du = 1, \quad h > 0, \, h \ \text{continuous}.$$

Then where $\alpha = c^2$

$$\|f - P_{m, T_n} f\|_{\tilde{Q}_1}^2 = \frac{1}{n^{2m}} \left( \frac{(m!)^2}{(2m)! (2m + 1)!} \int_0^1 \rho^2(s) \alpha(s) \, ds \right) + o\left( \frac{1}{n^{2m}} \right).$$

**Remark.** The hypotheses of the Theorem do not include $I + A$ invertible. On the other hand, if $I + A$ is invertible then condition (iv) is equivalent to $\gamma_t \in \mathcal{H}_{K_0}$, the RKHS with RK

$$K_0(s, t) = \int_0^{\min(s, t)} c^2(u) \, du,$$
where $\mathcal{H}_{K_0} = \{ f : f(0) = 0, f \text{ abs. cont.}, f'/c \in \mathcal{L}_2 \}$. Thus if $I + A$ is invertible and (ii) holds then (iv) is equivalent to

$$\int_0^1 \left( \frac{1}{c(t)} A(t, \eta) \right)^2 d\eta < M < \infty.$$  

Condition (iv) is similar to a condition used in [11]. This condition is used in the proof of Lemma 1 to follow, and we see no way to eliminate it there.

**Proof.** The proof below follows closely along the lines of the proof of Theorem 1 of [13], generalized with the aid of [16].

The proof begins with Lemma 1.

**Lemma 1.** Let

$$K_0(s, t) = \int_0^{\min(s, t)} c^2(u) \, du$$

$$K_1(s, t) = \int_0^{\min(s, t)} c^2(u) \, du + \int_0^s \int_0^t c(u) A(u, v) c(v) \, du \, dv$$

$$f_0(t) = \int_0^t K_0(t, u) \rho(u) \, du$$

$$f_1(t) = \int_0^t K_1(t, u) \rho(u) \, du$$

where $\int_0^s \int_0^t A(u, v) \, du \, dv < \infty$, where $c, \rho > 0$, continuous, $c', \rho'$ bounded. Let $\mathcal{H}_i$, $i = 0, 1$, be the RKHS's with reproducing kernels $K_i$, $i = 0, 1$, and inner products $\langle \cdot, \cdot \rangle_{K_0}$ and $\langle \cdot, \cdot \rangle_{K_1}$ respectively. Suppose further that, for each $t$, the function $\gamma_t$ defined by

$$\gamma_t(s) = \int_s^1 \frac{1}{c(t)} A(t, \eta) c(\eta) \, d\eta$$

satisfies

$$(3.1) \quad \gamma_t \in \mathcal{H}_i, \quad ||\gamma_t||_{K_i} \leq M_i < \infty, \quad t \in [0, 1].$$

Then, there exists an $\varepsilon$ independent of $\rho$ such that, for sufficiently large $n$,

$$(3.2) \quad 1 - \varepsilon \Delta \leq \frac{||f_1 - P_{T_n} f_1||_K}{||f_0 - P_{T_n} f_0||_K} \leq 1 + \varepsilon \Delta$$

where

$$\Delta = \max_t |t_{i+1} - t_i|.$$  

Here, for $i = 0, 1$, $P_{T_n} f_i$ is the projection of $f_i$ in $\mathcal{H}_i$ onto the subspace of $\mathcal{H}_i$ spanned by $\{K_{it}, t \in T_\delta\}$, where $K_{it}(t') = K_i(t, t')$.

**Proof.** For $i = 0, 1$,

$$\langle f_i - P_{T_n} f_i, f_i - P_{T_n} f_i \rangle_{K_i} = \langle f_i, f_i - P_{T_n} f_i \rangle_{K_i} = \int_0^1 \rho(u) (f_i(u) - P_{T_n} f_i(u)) \, du.$$  

Then

$$||f_0 - P_{T_n} f_0||_{K_0}^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u) (f_0(u) - P_{T_n} f_0(u)) \, du$$

$$= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u) \, du \int_{t_i}^{t_{i+1}} B_i(u, v) \rho(v) \, dv$$

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where \( t_0 \equiv 0, t_n \equiv 1 \) and, according to [16] \( B_t(u, v) \) is, for \( u, v \in [t_i, t_{i+1}] \), the Green’s function for the differential operator \( L_m^*L_m = g \) with boundary conditions \( f(t_i) = f(t_{i+1}) = 0 \),

\[
(L_m^*L_m f)(t) = \frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} f(t).
\]

Similarly,

\[
||f_1 - P_{\tau_n}f_1||_{K_1}^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)(f_1(u) - P_{\tau_n}f_1(u)) \, du.
\]

Since \( f_1(u) - P_{\tau_n}f_1(u) = 0 \) for \( t = t_0, t_1, \ldots, t_n \), and \( f_1 - P_{\tau_n}f_1 \in D[0, 1] \), we may write

\[
(f_1 - P_{\tau_n}f_1)(u) = \int_{t_i}^{t_{i+1}} B_t(u, v) \frac{d}{dv} \frac{1}{c^2(v)} \frac{d}{dv} (f_1(v) - P_{\tau_n}f_1(v)) \, dv,
\]

where \( B_t \) is as before. But, since

\[
f_1(t) = \int_0^t K_0(t, u) \rho(u) \, du + \int_0^t \rho(u) \, du \int_0^t \int_0^1 c(\xi) A(\xi, \eta) c(\eta) \, d\xi \, d\eta,
\]

then

\[
\frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} f_1(t) = \rho(t) + \int_0^t \rho(u) \, du \int_0^t \frac{\partial}{\partial t} \frac{1}{c(t)} A(t, \eta) c(\eta) \, d\eta
\]

\[
= \rho(t) + \int_0^t \rho(u) \gamma_i(u) \, du.
\]

By our assumption, \( \gamma_t \in \mathcal{H}_{K_1} \), so that (3.5) becomes

\[
\frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} f_1(t) = \rho(t) + \int_0^t \rho(u) \gamma_i(u) = \rho(t) + \langle \gamma_t, f_1 \rangle_{K_1}.
\]

Also

\[
(P_{\tau_n}f_1)(t) = (K_1(t, t_1), K_1(t, t_2), \ldots, K_1(t, t_n))K_{1,n}^{-1}(f_1(t_1), f_1(t_2), \ldots, f_1(t_n))'
\]

where \( K_{1,n} \) is the \( n \times n \) matrix with \( ij \)th entry \( K_1(t_i, t_j) \). Now, for \( t \neq t_i \),

\[
\frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} K_1(t, t_i) = \frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} \left[ \int_0^t \int_0^t c(\xi) A(\xi, n) c(\eta) \, d\xi \, d\eta \right] = \gamma_i(t_i)
\]

so that, for each fixed \( t \in T_n \),

\[
\frac{d}{dt} \frac{1}{c^2(t)} \frac{d}{dt} (P_{\tau_n}f_1(t))
\]

\[
= \langle \gamma_i(t_i), \gamma_i(t_2), \ldots, \gamma_i(t_n) \rangle K_{1,n}^{-1}(f_1(t_1), f_1(t_2), \ldots, f_1(t_n))'
\]

\[
= \langle \gamma_i, P_{\tau_n}f_1 \rangle_{K_1}.
\]

Thus, by (3.3), (3.4), (3.6), (3.7),

\[
||f_1 - P_{\tau_n}f_1||_{K_1}^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u) \, du \int_{t_i}^{t_{i+1}} B_t(u, v) \rho(v) \, dv
\]

\[
+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u) \, du \int_{t_i}^{t_{i+1}} B_1(u, v) \langle \gamma_i, f_1 - P_{\tau_n}f_1 \rangle_{K_1} \, dv.
\]
Now \( \rho \) and \( B_i \) are nonnegative, so we may write
\[
|\int \int_{t_i}^{t_{i+1}} \rho (u) \, du \int B_i(u, v) \langle \gamma_v, f_i - P_{T_n} f_i \rangle_{K_1} \, dv| \\
\leq \int \int_{t_i}^{t_{i+1}} \rho (u) \, du \int B_i(u, v) \, dv \times M_i \| f_i - P_{T_n} f_i \|_{K_1} ,
\]
where \( M_i \) is defined in (3.1).
Now, letting
\[
\xi_0 (t) = \delta K_0 (t, u) \, du
\]
it may be shown that
\[
(3.8) \quad \sum_{i=0}^{n-1} \int \int_{t_i}^{t_{i+1}} \rho (u) \, du \int B_i(u, v) \, dv = \left\langle f_0 - P_{T_n} f_0, \xi_0 - P_{T_n} \xi_0 \right\rangle_{K_0} .
\]
By (1.4),
\[
\| \xi_0 - P_{T_n} \xi_0 \|_{K_0} = M_2 \left( \frac{1}{n} \left( 1 + o \left( \frac{1}{n} \right) \right) \right)
\]
for appropriately chosen \( M_2 \).
Thus
\[
\| f_1 - P_{T_n} f_1 \|_{K_1} = \| f_0 - P_{T_n} f_0 \|_{K_0} + \theta \frac{M_2}{n} \| f_1 - P_{T_n} f_1 \|_{K_1} \| f_0 - P_{T_n} f_0 \|_{K_0}
\]
for some \( \theta \) with \( |\theta| < 1 \) and \( M_3 = M_1 M_2 \), and so
\[
\| f_1 - P_{T_n} f_1 \|_{K_1} = 1 + \theta \frac{M_2}{n} \left( 1 + o \left( \frac{1}{n} \right) \right) .
\]
Since \( 1/n \leq \Delta \), the Lemma is proved.

**Lemma 2.** For \( m \geq 2 \) let
\[
Q_i (s, t) = \delta K_i (u, v) \, du \, dv i = 1, 2
\]
where \( K_i \), \( i = 0, 1 \) are as Lemma 1. Let
\[
f_i (t) = \int Q_i (t, u) \rho (u) \, du ,
\]
where \( f_i \) is the projection of \( f_i \) in \( H_{Q_i} \), \( i = 0, 1 \)
onto the subspace of (1.1) with \( Q = Q_i \).

Then, there exists an \( \epsilon \) independent of \( \rho \) such that, for sufficiently large \( n \),
\[
1 - \epsilon \Delta \leq \frac{\| f_1 - P_{m,T_n} f_1 \|_{Q_1}^2}{\| f_0 - P_{m,T_n} f_0 \|_{Q_0}^2} \leq 1 + \epsilon \Delta .
\]
Here \( P_{m,T_n} f_i \) is the projection of \( f_i \) in \( H_{Q_i} \), \( i = 0, 1 \) onto the subspace of (1.1) with \( Q = Q_i \).

The proof of this Lemma is contained within the proof of Theorem 1 of [13], page 2065 Equations (2.28)–(2.31), where it is shown that (3.2) implies (3.9).

Theorem now follows by using the proof of Lemma 3 of [16] (where only condition ii on \( c \) is needed for \( Q = Q_0 \) to show that
\[
\| f_0 - P_{m,T_n} f_0 \|_{Q_0}^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(m+1)!} \int \int \rho (s) \, ds + o \left( \frac{1}{n^{2m}} \right) .
\]

\* Equation (3.8) may be checked by following the argument of Lemma 1 of [16]; see equations (3.4), (3.5) and (3.22). Equation (3.4a) there should read \( f(t) = EX(t) \int \mathcal{X}(u) \rho (u) \, du \).
REFERENCES