THEORY OF PROBABILITY
Volume XVIII AND ITS APPLICATIONS Number 1
1973

ON ABSOLUTE CONTINUITY OF MEASURES CORRESPONDING TO HOMOGENEOUS GAUSSIAN FIELDS

A. V. SKOROKHOD AND M. I. YADRENKO

(*Translated by K. Durr*)

General questions of absolute continuity and singularity of Gaussian measures have been considered in works of Ya. Gaek [1], J. Feldman [2] and Yu. A. Rozanov [3]. However, in considering concrete Gaussian measures it is desirable to be able to answer these questions using only the defining characteristics of the corresponding processes.

As is known, to solve the problem of absolute continuity and to find the density it is necessary to solve a certain operator equation, which for ordinary processes leads to a Fredholm integral equation of the first type. The existence of a solution of this equation ensures absolute continuity. But the question of the existence of solutions of such equations is very complex. Hence there arises the problem: to find conditions of absolute continuity of measures which do not involve the existence of a solution of the corresponding equations.

For stationary processes, several conditions expressed in terms of correlation functions or spectral densities have been given by Rozanov [4], [3]. Other general conditions appear in the summary report of I. I. Gikhman and A. V. Skorokhod [6], as well as in the book by the same authors [7] (Chapter 7, § 5).

In the present paper, analogous conditions using only spectral functions and densities are found for homogeneous Gaussian fields. The authors have restricted themselves only to the case when the means of the Gaussian fields are equal to zero and the correlation functions differ. The case of identical correlation functions and distinct means is studied by M. I. Yadrenko in [8]. Combining the results of [8] with those of this paper one can obtain conditions of absolute continuity of homogeneous fields for distinct means and correlation functions. To be especially noted is the case of isotropic Gaussian fields which are considered separately.

Conditions of absolute continuity and singularity of measures corresponding to Gaussian random fields have not yet been studied sufficiently. In this connection note the works of Z. S. Zerakidze [9], [10], G. M. Molchan and Yu. I. Golosov [11].

1. Notation, Statement of the Problem, Auxiliary Results

Let $\mathbf{R}^n = \{\mathbf{t}:\mathbf{t} = (t_1, \cdots, t_n)\}$ be n-dimensional Euclidean space, and let $\mathscr{D}$ be a closed bounded region in $\mathbf{R}^n$; if $\mathbf{T} = (T_1, \cdots, T_n)$, then $\Pi_{\mathbf{T}}$ is the parallelopiped

$$\Pi_{\mathbf{T}} = \{\mathbf{t}: -T_k \leqq t_k \leqq T_k, k = 1, \cdots, n\};$$

$L_2(\mathscr{D})$ is the space of square-integrable functions on $\mathscr{D}$, and if $x(\mathbf{t}) \in L_2(\mathscr{D})$, $y(\mathbf{t}) \in L_2(\mathscr{D})$, then

$$(x(\mathbf{t}), y(\mathbf{t})) = \int_{\mathscr{D}} x(\mathbf{t})\overline{y(\mathbf{t})}\, d\mathbf{t};$$

if $\mathbf{t} \in R^n$, $\mathbf{s} \in R^n$, then

$$(\mathbf{t}, \mathbf{s}) = \sum_{k=1}^{n} t_k s_k.$$

Let $\xi_1(\mathbf{t})$ and $\xi_2(\mathbf{t})$ be Gaussian homogeneous random fields on $\mathbf{R}^n$ with zero mathematical expectations and correlation functions $R_1(\mathbf{t} - \mathbf{s})$ and $R_2(\mathbf{t} - \mathbf{s})$. As is known,

$$R_j(\mathbf{t} - \mathbf{s}) = \int_{\mathbf{R}^n} e^{i(\lambda, \mathbf{t}-\mathbf{s})} F_j(d\boldsymbol{\lambda}), \qquad j = 1, 2, \tag{1}$$

where $F_j(\cdot)$ is a finite measure on the σ-algebra $\mathbf{B}$ of Borel sets in $\mathbf{R}^n$ (the spectral measure of $\xi_j(\mathbf{t})$). If $F_j(\cdot)$ is absolutely continuous relative to Lebesgue measure, then $dF_j/d\lambda = f_j(\boldsymbol{\lambda})$ is the spectral density of $\xi_j(\mathbf{t})$.

The random field $\xi_j(\mathbf{t})$ has the representation

$$\xi_j(\mathbf{t}) = \int_{\mathbf{R}^n} e^{i(\lambda, \mathbf{t})} Z_j(d\boldsymbol{\lambda}), \tag{2}$$

where $Z_j(\cdot)$ is a random additive set function on $\mathbf{B}$ such that

$$\mathbf{E}Z_j(S_1)\overline{Z_j(S_2)} = F_j(S_1 \cap S_2), \qquad S_1 \in \mathbf{B}, S_2 \in \mathbf{B}. \tag{3}$$

We shall also consider homogeneous and isotropic random fields. The correlation function of such a field $R_j(\rho)$ depends only on the distance between the points $\mathbf{t}$ and $\mathbf{s}$, the spectral measure $F_j(\cdot)$ is invariant under rotations around the origin and the representation (1) takes on the form

$$R_j(\rho) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right) \int_0^\infty \frac{J_{(n-2)/2}(\lambda\rho)}{(\lambda\rho)^{(n-2)/2}}\, d\Phi_j(\lambda), \tag{4}$$

where $\Phi_j(\lambda) = \int_{|\lambda| < \lambda} F(d\boldsymbol{\lambda})$ is a bounded nondecreasing function, $J_\nu(x)$ is the Bessel function of first kind of ν-th order. A homogeneous and isotropic random field $\xi_j(\mathbf{t})$ has the form (see [12], [13]):

$$\xi_j(\mathbf{t}) = c_n \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(r, \theta_1, \cdots, \theta_{n-2}, \varphi) \int_0^\infty \frac{J_{m+(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}} Z_m^l(d\lambda), \tag{5}$$

where $(r, \theta_1, \cdots, \theta_{n-2}, \varphi)$ are the spherical coordinates of $\mathbf{t}$, and $S_m^l(\theta_1, \cdots, \theta_{n-2}, \varphi)$ are the orthonormal spherical harmonics of degree m (cf. [14]), $c_n^2 = 2^{n-1}\Gamma(n/2)\pi^{n/2}$, and

$$h(m, n) = (2m + n - 2)\frac{(m + n - 3)!}{(n - 2)!\,m!}$$

is the number of such harmonics, $Z_m^l(\cdot)$ is a sequence of additive random set functions defined on Borel sets in $(0, +\infty)$ such that

$$\mathbf{E}Z_m^l(S) = 0, \qquad \mathbf{E}Z_m^l(S_1)\overline{Z_p^q(S_2)} = \delta_m^p\,\delta_l^q\Phi_j(S_1 \cap S_2). \tag{6}$$

The random fields $\xi_j(\mathbf{t})$, $j = 1, 2$, induce, on the Hilbert space $L_2(\mathscr{D})$, Gaussian measures $\mu_j^{\mathscr{D}}$ with zero means and correlation operators

$$\mathscr{B}_j x(\mathbf{t}) = \int_{\mathscr{D}} R_j(\mathbf{t} - \mathbf{s})\overline{x(\mathbf{s})}\,d\mathbf{s}. \tag{7}$$

We shall be interested in conditions of absolute continuity of the measures $\mu_2^{\mathscr{D}}$ and $\mu_1^{\mathscr{D}}$, expressed in terms of the spectral measures and spectral densities as well as in the calculation of $d\mu_2^{\mathscr{D}}/d\mu_1^{\mathscr{D}}$.

In the sequel we shall use general conditions of absolute continuity of measures in Hilbert space. Let us formulate them (the proofs are given in [7]).

Let μ_1 and μ_2 be Gaussian measures defined on the σ-algebra $\mathscr{L}$ of Borel sets in Hilbert space X, where the mean values of μ_1 and μ_2 are equal to zero and the correlation operators are $\mathscr{B}_1$ and $\mathscr{B}_2$.

Assertion A. *The measures μ_1 and μ_2 are absolutely continuous if and only if*

1) *the operator $D = \mathscr{B}_2^{-1/2}\mathscr{B}_1\mathscr{B}_2^{-1/2} - I$ is a Hilbert–Schmidt operator;*
2) *the eigenvalues δ_k of D are greater than -1.*

If μ_1 and μ_2 are absolutely continuous then formula

$$\frac{d\mu_2}{d\mu_1} = \exp\left\{-\frac{1}{2}\sum_{k=1}^{\infty}\left[(\mathscr{B}_2^{-1/2}x, e_k)^2\frac{\delta_k}{1 + \delta_k} - \log(1 + \lambda_k)\right]\right\} \tag{8}$$

holds, where e_k are the eigenvectors of D corresponding to the δ_k.

Assertion B. *If there is a bounded operator V satisfying the condition*

$$V\mathscr{B}_2 = \mathscr{B}_1 - \mathscr{B}_2,$$

$\mathrm{Sp}\, V^*V < \infty$, and -1 is not in the spectrum of V, then μ_1 and μ_2 are absolutely continuous.

If the assumptions of Assertion B hold and $\mathrm{Sp}\, V$ is defined (i.e., the series $\Sigma\,(Ve_k, e_k)$ converges in each orthonormal basis), then

$$\frac{d\mu_2}{d\mu_1} = \sqrt{\det(I + V)}\exp\{-\tfrac{1}{2}(\mathscr{B}^{-1}Vx, x)\}. \tag{9}$$

As is known, if the Gaussian measure μ_2 is absolutely continuous relative to μ_1 ($\mu_2 < \mu_1$), then μ_1 and μ_2 are equivalent ($\mu_1 \sim \mu_2$), i.e., $\mu_1 < \mu_2$.

Let $\mathscr{W}_{\mathscr{D}}$ be the class of Fourier transformations of functions in $L_2(\mathscr{D})$ which are zero off $\mathscr{D}$. By a theorem of Pólya–Plancherel (cf. [15]) the function $g(\boldsymbol{\lambda}) \in \mathscr{W}_{\Pi_{\mathbf{T}}}$ if and only if it can be extended in complex n-dimensional space to an entire function of exponential type $\mathbf{T} = (T_1, \cdots, T_n)$ and is square-integrable on $\mathbf{R}^n$. By $\mathscr{W}_{\mathscr{D}}(F_1)$ we denote the closure of $\mathscr{W}_{\mathscr{D}}$ in the metric

$$\|g(\boldsymbol{\lambda})\|_{F_1}^2 = \int |g(\boldsymbol{\lambda})|^2 F_1(d\boldsymbol{\lambda}).$$

Let $\mathscr{W}_{\mathscr{D}}^2$ be the space of functions $b(\boldsymbol{\lambda}, \boldsymbol{\mu})$ which can be represented in the form

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \int_{\mathscr{D}} \int_{\mathscr{D}} \exp\{i(\mathbf{t}, \boldsymbol{\lambda}) - i(\mathbf{s}, \boldsymbol{\mu})\} \varphi(\mathbf{t}, \mathbf{s})\, d\mathbf{t}\, d\mathbf{s},$$

where $\varphi(\mathbf{t}, \mathbf{s}) \in L_2(\mathscr{D} \times \mathscr{D})$, and $\mathscr{W}_{\mathscr{D}}^2(F_1)$ is the closure of $\mathscr{W}_{\mathscr{D}}^2$ in the metric generated by the scalar product

$$(b_1, b_2) = \int \int b_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) \overline{b_2(\boldsymbol{\lambda}, \boldsymbol{\mu})} F_1(d\boldsymbol{\lambda}) F_1(d\boldsymbol{\mu}).$$

2. Conditions of Absolute Continuity of Measures Corresponding to Random Homogeneous Fields

Theorem 1. *The measures $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are absolutely continuous if and only if there is a function $b(\boldsymbol{\lambda}, \boldsymbol{\mu})$ in $\mathscr{W}_{\mathscr{D}}^2(F_1)$ such that*

$$R_2(\mathbf{t} - \mathbf{s}) - R_1(\mathbf{t} - \mathbf{s}) = \int \int \exp\{-i(\boldsymbol{\lambda}, \mathbf{t}) + i(\boldsymbol{\mu}, \mathbf{s})\} b(\boldsymbol{\lambda}, \boldsymbol{\mu}) F_1(d\boldsymbol{\lambda}) F_1(d\boldsymbol{\mu}), \quad (\mathbf{t}, \mathbf{s}) \in \mathscr{D} \times \mathscr{D}. \tag{10}$$

Here

$$\frac{d\mu_2}{d\mu_1}(\xi_1(\cdot)) = \exp\left\{\int \int \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}) Z_1(d\boldsymbol{\lambda}) Z_1(d\boldsymbol{\mu}) + c\right\}, \tag{11}$$

where $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is related to $b(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by the relation:

$$\int \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}) b(\boldsymbol{\mu}, \mathbf{v}) F_1(d\boldsymbol{\mu}) = b(\boldsymbol{\lambda}, \mathbf{v}) - \theta(\boldsymbol{\lambda}, \mathbf{v}) \tag{12}$$

and

$$c = -\log \mathbf{E} \exp\{\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}) Z_1(d\boldsymbol{\lambda}) Z_1(d\boldsymbol{\mu})\}. \tag{13}$$

PROOF. NECESSITY. Let μ_1 and μ_2 be equivalent. Then the spaces $L(\mu_1)$ and $L(\mu_2)$ of linear functionals which are measurable with respect to μ_1 and μ_2 are the same. Each measurable linear functional for a homogeneous random field has the form $l(\xi_j(\mathbf{t}), \mathbf{t} \in \mathscr{D}) = \int g(\boldsymbol{\lambda}) Z_j(d\boldsymbol{\lambda})$, where $g(\boldsymbol{\lambda}) \in \mathscr{W}_{\mathscr{D}}(F_1)$. From general results on absolute continuity (cf. [7]) in Hilbert space it follows that if μ_1 and μ_2 are equivalent, then one can construct a sequence of measurable linear functionals l_k forming a complete orthogonal system in $L(\mu_1)$ and $L(\mu_2)$ (these are the functionals $(\mathscr{B}_2^{-1/2}x, e_k)$). Let $l_k = \int g_k(\boldsymbol{\lambda})Z_k(d\boldsymbol{\lambda})$. From the orthogonality of the l_k with respect to μ_1 and μ_2 it follows that

$$\mathbf{E}\int g_k(\boldsymbol{\lambda})Z_j(d\boldsymbol{\lambda})\overline{\int g_m(\boldsymbol{\lambda})Z_j(d\boldsymbol{\lambda})} = \int g_k(\boldsymbol{\lambda})\overline{g_m(\boldsymbol{\lambda})}F_j(d\boldsymbol{\lambda}) = 0$$

for $k \neq m$. We shall norm $g_k(\boldsymbol{\lambda})$ so that

$$\int |g_k(\boldsymbol{\lambda})|^2 F_1(d\boldsymbol{\lambda}) = 1, \qquad \int |g_k(\boldsymbol{\lambda})|^2 F_2(d\boldsymbol{\lambda}) = 1 + c_k.$$

From what was said above regarding the l_k, it follows here that $\Sigma c_k^2 < \infty$.

Let us set

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{\infty} c_k g_k(\boldsymbol{\lambda})g_k(\boldsymbol{\mu}).$$

Then $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}_{\mathscr{D}}^2(F_1)$, and one can see that $b(\boldsymbol{\lambda}, \boldsymbol{\mu})$ satisfies (10). To this end consider the function

$$\gamma(\mathbf{t}, \mathbf{s}) = \int\int \exp\{-i(\boldsymbol{\lambda}, \mathbf{t}) + i(\boldsymbol{\mu}, \mathbf{s})\}b(\boldsymbol{\lambda}, \boldsymbol{\mu})F_1(d\boldsymbol{\lambda})F_1(d\boldsymbol{\mu}) + R_1(\mathbf{t} - \mathbf{s}) - R_2(\mathbf{t} - \mathbf{s})$$

and let us show that $\gamma(\mathbf{t}, \mathbf{s}) = 0$. Let $\varphi(\mathbf{t})$ be any function in $L_2(\mathscr{D})$ and

$$u(\boldsymbol{\lambda}) = \int_{\mathscr{D}} \exp\{-i(\boldsymbol{\lambda}, \mathbf{t})\}\varphi(\mathbf{t})\, d\mathbf{t}.$$

Then using Parseval's equality we obtain

$$\begin{aligned}\int\int_{\mathscr{D}\times\mathscr{D}} \gamma(\mathbf{t}, \mathbf{s})\varphi(\mathbf{t})\overline{\varphi(\mathbf{s})}\, d\mathbf{t}\, d\mathbf{s} &= \int\int u(\boldsymbol{\lambda})\overline{u(\boldsymbol{\mu})}b(\boldsymbol{\lambda}, \boldsymbol{\mu})F_1(d\boldsymbol{\lambda})F_1(d\boldsymbol{\mu}) \\ &\quad + \int |u(\boldsymbol{\lambda})|^2 F_1(d\boldsymbol{\lambda}) - \int |u(\boldsymbol{\lambda})|^2 F_2(d\boldsymbol{\lambda}) \\ &= \sum_{k=1}^{\infty}\left(1 + c_k - \int |g_k(\boldsymbol{\lambda})|^2 F_2(d\boldsymbol{\lambda})\right) \\ &\quad \times \int u(\boldsymbol{\lambda})\overline{g_k(\boldsymbol{\lambda})}F_1(d\boldsymbol{\lambda}) = 0.\end{aligned}$$

Thus $\gamma(\mathbf{t}, \mathbf{s}) = 0$.

SUFFICIENCY. Let $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}_{\mathscr{D}}^2(F_1)$ and satisfy (10). Consider the integral operator

$$Vg(\boldsymbol{\mu}) = \int b(\boldsymbol{\lambda}, \boldsymbol{\mu})g(\boldsymbol{\lambda})F_1(d\boldsymbol{\lambda}). \tag{14}$$

If $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}_{\mathscr{D}}^2(F_1)$, then this operator maps $\mathscr{W}_{\mathscr{D}}(F_1)$ into itself and is Hilbert–Schmidt. Let $g_k(\boldsymbol{\lambda})$ denote a complete orthonormal sequence of eigenfunctions of V and let λ_k denote the corresponding eigenvalues. Then

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{\infty} \lambda_k \overline{g_k(\boldsymbol{\lambda})} g_k(\boldsymbol{\mu}), \qquad \sum_{k=1}^{\infty} \lambda_k^2 < \infty.$$

We shall prove that $\{g_k(\boldsymbol{\lambda})\}$ is orthogonal not only in $\mathscr{W}_{\mathscr{D}}(F_1)$ but also in $\mathscr{W}_{\mathscr{D}}(F_2)$. Let $\varphi_{kn}(\mathbf{t})$ be a sequence of functions in $L_2(\mathscr{D})$ such that

$$u_{kn}(\mathbf{t}) = \int_{\mathscr{D}} \exp\{-i(\boldsymbol{\lambda}, \mathbf{t})\} \varphi_{kn}(\mathbf{t})\, d\mathbf{t} \to g_k(\boldsymbol{\lambda})$$

as $n \to \infty$ in the sense of convergence of $\mathscr{W}_{\mathscr{D}}(F_1)$. Then

$$\iint_{\mathscr{D} \times \mathscr{D}} R_2(\mathbf{t} - \mathbf{s}) \varphi_{kn}(\mathbf{t}) \overline{\varphi_{jn}(\mathbf{s})}\, d\mathbf{t}\, d\mathbf{s} - \iint_{\mathscr{D} \times \mathscr{D}} R_1(\mathbf{t} - \mathbf{s}) \varphi_{kn}(\mathbf{t}) \overline{\varphi_{jn}(\mathbf{s})}$$

$$= \iint b(\boldsymbol{\lambda}, \boldsymbol{\mu}) u_{jn}(\boldsymbol{\lambda}) \overline{u_{kn}(\boldsymbol{\mu})} F_1(d\boldsymbol{\lambda}) F_1(d\boldsymbol{\mu}).$$

Passing to the limit as $n \to \infty$ we obtain

$$\int g_k(\boldsymbol{\lambda}) \overline{g_j(\boldsymbol{\lambda})} F_2(d\boldsymbol{\lambda}) - \int g_k(\boldsymbol{\lambda}) \overline{g_j(\boldsymbol{\lambda})} F_1(d\boldsymbol{\lambda})$$

$$= \iint b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \overline{g_k(\boldsymbol{\lambda})} g_j(\boldsymbol{\mu}) F_1(d\boldsymbol{\lambda}) F_1(d\boldsymbol{\mu}) = 0.$$

From (10) it follows that $b(\boldsymbol{\lambda}, \boldsymbol{\mu})$ can be chosen so that $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = b(-\boldsymbol{\lambda}, -\boldsymbol{\mu})$, and then the $g_k(\boldsymbol{\lambda})$ can be chosen so that $g_k(\boldsymbol{\lambda}) = \overline{g_k(-\boldsymbol{\lambda})}$. Under this condition $\int g_k(\boldsymbol{\lambda}) Z_j(d\boldsymbol{\lambda})$ will be a sequence of real linear orthogonal functionals of the random fields $\xi_j(\mathbf{t})$.

By Assertion A, $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are equivalent to

$$\frac{d\mu_2^{\mathscr{D}}}{d\mu_1^{\mathscr{D}}}(\xi_1(\cdot)) = \exp\left\{-\frac{1}{2} \sum_{k=1}^{\infty} \left[\frac{\lambda_k}{1 + \lambda_k} \left| \int g_k(\boldsymbol{\lambda}) Z_1(d\boldsymbol{\lambda}) \right|^2 - \log(1 + \lambda_k)\right]\right\}.$$

It remains only to note that

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{1 + \lambda_k} \left[\left| \int g_k(\boldsymbol{\lambda}) Z_1(d\boldsymbol{\lambda}) \right|^2 - 1 \right] = \iint \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}) Z_1(d\boldsymbol{\lambda}) Z_1(d\boldsymbol{\mu}),$$

where

$$\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{\infty} \frac{\lambda_k}{1 + \lambda_k} \overline{g_k(\boldsymbol{\lambda})} g_k(\boldsymbol{\mu}).$$

Theorem 1 is proved.

Corollary 1. *If there exists a function $c(\mathbf{u}, \mathbf{v}) \in L(\mathscr{D} \times \mathscr{D})$, satisfying the equation*

$$\Delta(\mathbf{t} - \mathbf{s}) = R_2(\mathbf{t} - \mathbf{s}) - R_1(\mathbf{t} - \mathbf{s}) = \iint_{\mathscr{D} \times \mathscr{D}} R_1(\mathbf{u} - \mathbf{t}) c(\mathbf{u}, \mathbf{v}) R_1(\mathbf{s} - \mathbf{v})\, d\mathbf{u}\, d\mathbf{v}, \tag{15}$$

$$\mathbf{t} \in \mathscr{D}, \mathbf{s} \in \mathscr{D},$$

then $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are equivalent.

Indeed, (10) follows from (15) if we set

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \iint_{\mathscr{D} \times \mathscr{D}} \exp\{i(\mathbf{u}, \boldsymbol{\lambda}) - i(\mathbf{v}, \boldsymbol{\mu})\} c(\mathbf{u}, \mathbf{v})\, d\mathbf{u}\, d\mathbf{v},$$

and $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}^2_{\mathscr{D}}(F_1)$.

Let us suppose that $\zeta_1(\mathbf{t})$ has spectral density $f_1(\boldsymbol{\lambda})$ and that $f_1(\boldsymbol{\lambda})$ is bounded on $\mathbf{R}^n$. Then from the fact that $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}^2_{\mathscr{D}}(F_1)$ it follows that $b(\boldsymbol{\lambda}, \boldsymbol{\mu}) f_1(\boldsymbol{\lambda}) f_1(\boldsymbol{\mu}) \in L_2(\mathbf{R}^n \times \mathbf{R}^n)$. Using this fact it is not hard to obtain, by means of arguments analogous to those used in proof of Theorems 11 and 12 of Chapter III in [16], the following result due to Zerakidze [10].

Theorem 2. *The measures $\boldsymbol{\mu}_1^{\mathscr{D}}$ and $\boldsymbol{\mu}_2^{\mathscr{D}}$ are equivalent if and only if the function*

$$\Delta(\mathbf{t} - \mathbf{s}) = R_2(\mathbf{t} - \mathbf{s}) - R_1(\mathbf{t} - \mathbf{s}), \qquad (\mathbf{t}, \mathbf{s}) \in \mathscr{D} \times \mathscr{D},$$

can be extended to a function which is square-integrable on $\mathbf{R}^n \times \mathbf{R}^n$ and whose Fourier transformation $\varphi(\boldsymbol{\lambda}, \boldsymbol{\mu})$ satisfies the condition

$$\iint \frac{|\varphi(\boldsymbol{\lambda}, \boldsymbol{\mu})|^2}{f_1(\boldsymbol{\lambda}) f_1(\boldsymbol{\mu})} d\boldsymbol{\lambda}\, d\boldsymbol{\mu} < \infty. \tag{16}$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a tuple of n non-negative integers, let

$$|\boldsymbol{\alpha}| = \sum_{k=1}^{n} \alpha_k, \quad D^\alpha f(x_1, \cdots, x_n) = \frac{\partial^{|\alpha|} f(x_1, \cdots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

and let $C_\infty(\mathscr{D})$ be the collection of infinitely differentiable functions whose supports are concentrated in $\mathscr{D}$.

We denote by $W_2^s(\mathscr{D})$ the closure of $C^\infty(\mathscr{D})$ in the metric

$$\|f\|_{W_2^s(\mathscr{D})} = \int_{\mathscr{D}} |f(\mathbf{x})|^2\, d\mathbf{x} + \sum_{|\alpha| = s} \int_{\mathscr{D}} |D^\alpha f(\mathbf{x})|^2\, d\mathbf{x}, \tag{17}$$

if s is an integer, and in the metric

$$\|f\|_{W_2^s(\mathscr{D})} = \int_{\mathscr{D}} |f(\mathbf{x})|^2\, d\mathbf{x} + \sum_{|\alpha| = s} \iint_{\mathscr{D} \times \mathscr{D}} \frac{|D^\alpha f(\mathbf{x}) - D^\alpha f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{n+2\gamma}} d\mathbf{x}\, d\mathbf{y}, \tag{18}$$

when $s = [s] + \gamma$, $0 < \gamma < 1$, i.e., the $W_2^s(\mathscr{D})$ are the known Sobolev classes. Suppose that the boundary of $\mathscr{D}$ satisfies conditions of continuation beyond $\mathscr{D}$, preserving the Sobolev class (cf. [19], p. 444).

Let

$$f_1(\boldsymbol{\lambda}) \asymp \left(1 + \sum_{k=1}^{n} \lambda_k^2\right)^{-l}. \tag{19}$$

From Theorem 2, using results of L. N. Slobodetskii [17] (see also [18]) we obtain

Theorem 3. *If condition (19) holds, then the measures $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are equivalent if and only if $(\Delta(\mathbf{t}) \in W_2^{2l}(\mathscr{D})$.*

For $n = 1$ this coincides with a result of Rozanov (Theorem 13, Chapter III of [16]) and in the general case generalizes a result of Zerakidze [10].

Now let us give sufficient conditions of equivalence of measures corresponding to homogeneous random fields under the assumption that there exist spectral densities. An important role in the sequel is played by the following lemma on orthogonal bases for $\mathscr{W}_{\mathscr{D}}(F_1)$.

Lemma. *Let $f_1(\boldsymbol{\lambda}) = |\varphi_0(\boldsymbol{\lambda})|^2$, $\varphi_0(\boldsymbol{\lambda}) \in \mathscr{W}_{\Pi_s}$ and let $\{g_k(\boldsymbol{\lambda})\}$ be an orthonormal basis in $\mathscr{W}_{\mathscr{D}}(F_1)$. Let $\Pi_{\mathbf{T}}, \mathbf{T} = (T_1, \cdots, T_n)$ be a parallelopiped enclosing the region $\mathscr{D}$. Then,*

$$\sum_{k=1}^{\infty} |g_k(\boldsymbol{\lambda})|^2 \leqq \frac{1}{\pi^n} \frac{\Pi_{k=1}^n (T_k + s_k)}{f_1(\boldsymbol{\lambda})}. \tag{20}$$

PROOF. Since $\mathscr{W}_{\mathscr{D}}$ is everywhere dense in $\mathscr{W}_{\mathscr{D}}(F_1)$, it suffices to show the inequality for the case when $g_k(\boldsymbol{\lambda}) \in \mathscr{W}_{\mathscr{D}}$. In this case, $q_k(\boldsymbol{\lambda})\varphi_0(\boldsymbol{\lambda}) \in \mathscr{W}_{\Pi_{\mathbf{T}+s}}$ and hence

$$g_k(\boldsymbol{\lambda})\varphi_0(\boldsymbol{\lambda}) = \int_{\Pi_{\mathbf{T}+s}} \exp\{-i(\boldsymbol{\lambda}, \mathbf{t})\}\psi_k(\mathbf{t})\, d\mathbf{t},$$

where $\psi_k(\mathbf{t}) \in L_2(\Pi_{\mathbf{T}+s})$. Since

$$\int g_k(\boldsymbol{\lambda})\varphi_0(\boldsymbol{\lambda})\overline{g_r(\boldsymbol{\lambda})\varphi_0(\boldsymbol{\lambda})}\, d\boldsymbol{\lambda} = \delta_k^r,$$

we have, by Parseval's equality,

$$\int_{\Pi_{\mathbf{T}+s}} \psi_k(\mathbf{t})\overline{\psi_r(\mathbf{t})}\, d\mathbf{t} = \frac{1}{(2\pi)^n}\delta_k^r.$$

Thus $\{2\pi^{n/2}\varphi_k(\mathbf{t})\}$ is an orthonormal system in $\Pi_{\mathbf{T}+s}$ and from Bessel's inequality we obtain

$$(2\pi)^n|\varphi_0(\boldsymbol{\lambda})|^2 \sum_{k=1}^{n} |g_k(\boldsymbol{\lambda})|^2 \leqq 2^n \prod_{k=1}^{n} (T_k + s_k),$$

and the assertion follows.

Theorem 4. *Let $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ be the measures corresponding to the homogeneous random fields $\xi_j(\mathbf{t})$ with $\mathbf{E}\xi_j(\mathbf{t}) = 0$ and spectral densities $f_j(\boldsymbol{\lambda})$, $\mathbf{t} \in \mathscr{D}$, $j = 1, 2$. If $f_1(\boldsymbol{\lambda}) \asymp |\varphi_0(\lambda)|^2$, where $\varphi_0(\lambda) \in \mathscr{W}_{\Pi_s}$ and*

$$\int \left[\frac{f_2(\boldsymbol{\lambda}) - f_1(\boldsymbol{\lambda})}{f_1(\boldsymbol{\lambda})}\right]^2 d\boldsymbol{\lambda} < \infty, \tag{21}$$

then $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are equivalent for every bounded region $\mathscr{D}$.

PROOF. Let us first prove this result under the assumption that $f_2(\boldsymbol{\lambda}) \geqq f_1(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \mathbf{R}^n$. Let

$$c_1|\varphi_0(\boldsymbol{\lambda})|^2 \leqq f_1(\lambda) \leqq c_2|\varphi_0(\boldsymbol{\lambda})|^2. \tag{22}$$

Consider the functions

$$\tilde{f}_1(\boldsymbol{\lambda}) = c_1|\varphi_0(\boldsymbol{\lambda})|^2, \qquad \tilde{f}_2(\boldsymbol{\lambda}) = c_1|\varphi_0(\boldsymbol{\lambda})|^2 + f_2(\boldsymbol{\lambda}) - f_1(\boldsymbol{\lambda}),$$
$$f_3(\boldsymbol{\lambda}) = f_1(\boldsymbol{\lambda}) - c_1|\varphi_0(\boldsymbol{\lambda})|^2.$$

We denote by $\tilde{\mu}_1^{\mathscr{D}}, \tilde{\mu}_2^{\mathscr{D}}, \mu_3^{\mathscr{D}}$ the measures induced by the Gaussian homogeneous random fields on $\mathscr{D}$ having, respectively, spectral densities $\tilde{f}_1(\lambda), \tilde{f}_2(\lambda), f_3(\lambda)$. Since $\mu_j^{\mathscr{D}} = \tilde{\mu}_j^{\mathscr{D}} * \mu_3^{\mathscr{D}}$, to prove the equivalence of $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ it suffices to show that $\tilde{\mu}_1^{\mathscr{D}} \sim \tilde{\mu}_2^{\mathscr{D}}$. Let us do this.

Let $F_j(\boldsymbol{\lambda})$ be the spectral function having spectral density $\tilde{f}_j(\boldsymbol{\lambda})$. Consider any orthonormal basis $\{g_k(\boldsymbol{\lambda})\}$ in $\mathscr{W}_{\mathscr{D}}(\tilde{F}_1)$. Let

$$h(\boldsymbol{\lambda}) = \frac{\tilde{f}_2(\boldsymbol{\lambda}) - \tilde{f}_1(\boldsymbol{\lambda})}{\tilde{f}_1(\boldsymbol{\lambda})}.$$

Then we have

$$\begin{aligned}
\sum_{k=1}^{\infty} [\|g_k\|_{\tilde{F}_2}^2 - \|g_k\|_{\tilde{F}_1}^2] &= \sum_{k=1}^{\infty} \int |g_k(\boldsymbol{\lambda})|^2 h(\boldsymbol{\lambda}) \tilde{f}_1(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \\
&\leqq \int \sum g_k^2(\boldsymbol{\lambda}) h^2(\boldsymbol{\lambda}) f_1(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \\
&\leqq \left(\prod_{k=1}^{n} \frac{T_k + s_k}{\pi} \right) \int \left[\frac{f_2(\boldsymbol{\lambda}) - f_1(\boldsymbol{\lambda})}{\tilde{f}_1(\boldsymbol{\lambda})} \right]^2 d\boldsymbol{\lambda} \\
&\leqq \left(\frac{c_2}{c_1} \right)^2 \left(\prod_{k=1}^{n} \frac{T_k + s_k}{\pi} \right) \int \left[\frac{f_2(\boldsymbol{\lambda}) - f_1(\boldsymbol{\lambda})}{f_1(\boldsymbol{\lambda})} \right]^2 d\boldsymbol{\lambda} < \infty.
\end{aligned}$$

Let V be a symmetric operator in $\mathscr{W}_{\mathscr{D}}(\tilde{F}_1)$ for which $(Vg, g) = \int |g(\boldsymbol{\lambda})|^2 \tilde{F}_2(d\lambda)$. We have shown that for every orthonormal basis $\{g_k(\boldsymbol{\lambda})\}$ in $\mathscr{W}_{\mathscr{D}}(\tilde{F}_1)$ the relation

$$\sum_{k=1}^{\infty} ([V - I]g_k, g_k)^2 < \infty$$

holds, i.e., $V - I$ is a Hilbert–Schmidt operator.

Let $\{g_k(\boldsymbol{\lambda})\}$ be a sequence of eigenfunctions of $V - I$ and let α_k be the corresponding eigenvalues. Then,

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1}^{\infty} \alpha_k \overline{g_k(\boldsymbol{\lambda})} g_k(\boldsymbol{\mu})$$

belongs to $\mathscr{W}_{\mathscr{D}}^2(\tilde{F}_1)$ since $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Let $\tilde{R}_j$ be the correlation function of the random field with spectral density $\tilde{f}_1(\boldsymbol{\lambda})$ and $\psi_{\mathbf{t}}(\boldsymbol{\lambda}) = \exp\{i(\boldsymbol{\lambda}, \mathbf{t})\}$. Then

$$\begin{aligned}
\tilde{R}_2(\mathbf{t} - \mathbf{s}) - \tilde{R}_1(\mathbf{t} - \mathbf{s}) &= ([V - I]\psi_{\mathbf{t}}, \psi_{\mathbf{s}}) \\
&= \sum_{k=1}^{\infty} \alpha_k (\psi_{\mathbf{t}}, g_k)(g_k, \psi_s) \\
&= \int \int \exp\{i(\mathbf{t}, \boldsymbol{\lambda}) - i(\mathbf{s}, \boldsymbol{\mu})\} b(\boldsymbol{\lambda}, \boldsymbol{\mu}) \tilde{F}_1(d\lambda) \tilde{F}_1(d\boldsymbol{\mu}).
\end{aligned}$$

By Theorem 1, $\tilde{\boldsymbol{\mu}}_2 \sim \tilde{\boldsymbol{\mu}}_1$. Thus it follows, as already noted, that $\boldsymbol{\mu}_2 \sim \boldsymbol{\mu}_1$.

Let us drop the assumption that $f_2(\lambda) > f_1(\lambda)$.

Consider the functions

$$\tilde{f}_1(\lambda) = c_1|\varphi_0(\lambda)|^2, \quad \tilde{f}_2(\lambda) = c_1|\varphi_0(\lambda)|^2 + \max(0, f_2 - f_1),$$

$$\tilde{f}_3(\lambda) = f_1(\lambda) - c_1|\varphi_0(\lambda)|^2, \quad f_4(\lambda) = \tilde{f}_2(\lambda) + \tilde{f}_3(\lambda).$$

Let $\tilde{\mu}_1^{\mathscr{D}}, \tilde{\mu}_2^{\mathscr{D}}, \tilde{\mu}_3^{\mathscr{D}}, \mu_4^{\mathscr{D}}$ be the measures corresponding to the homogeneous random fields with spectral densities $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, f_4$, respectively.

Note that $\tilde{\mu}_2^{\mathscr{D}} \sim \tilde{\mu}_1^{\mathscr{D}}$. In fact,

$$\int \left[\frac{\tilde{f}_2(\lambda) - \tilde{f}_1(\lambda)}{\tilde{f}_1(\lambda)}\right]^2 d\lambda \leqq \left(\frac{c_2}{c_1}\right)^2 \int \left[\frac{f_2(\lambda) - f_1(\lambda)}{f_1(\lambda)}\right]^2 d\lambda < \infty.$$

But $\mu_1^{\mathscr{D}} = \tilde{\mu}_1^{\mathscr{D}} * \tilde{\mu}_3^{\mathscr{D}}$, $\mu_4^{\mathscr{D}} = \tilde{\mu}_2^{\mathscr{D}} * \tilde{\mu}_3^{\mathscr{D}}$. Hence $\mu_1^{\mathscr{D}} \sim \mu_4^{\mathscr{D}}$. Moreover, $\mu_4^{\mathscr{D}} \sim \mu_2^{\mathscr{D}}$, since

$$\int \left[\frac{f_4(\lambda) - f_1(\lambda)}{f_4(\lambda)}\right]^2 d\lambda \leqq \int \left[\frac{f_2(\lambda) - f_1(\lambda)}{f_1(\lambda)}\right]^2 d\lambda < \infty.$$

Hence $\mu_1^{\mathscr{D}} \sim \mu_2^{\mathscr{D}}$.

Note that the theorem still holds even if (22) fails to hold on a set Δ of finite measure such that

$$\int_\Delta \left[\frac{f_i(\lambda)}{|\varphi_0(\lambda)|^2}\right]^2 d\lambda < \infty.$$

Indeed, in this case one can consider the measure $\mu_*^{\mathscr{D}}$ corresponding to the random field with spectral density

$$f_1^*(\lambda) = \begin{cases} f_1(\lambda), & \lambda \notin \Delta, \\ c_1|\varphi_0(\lambda)|^2, & \lambda \in \Delta. \end{cases}$$

The spectral density $f_1^*(\lambda)$ satisfies (22). Moreover, by assumption,

$$\int \left[\frac{f_i(\lambda) - f_1^*(\lambda)}{f_1^*(\lambda)}\right]^2 d\lambda < \infty, \qquad i = 1, 2.$$

Hence, $\mu_1^{\mathscr{D}} \sim \mu_1^{*\mathscr{D}}$, $\mu_2^{\mathscr{D}} \sim \mu_1^{*\mathscr{D}}$ and $\mu_1^{\mathscr{D}} \sim \mu_2^{\mathscr{D}}$.

Let us point out yet another sufficient condition for equivalence of $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ resulting from Assertion B.

Theorem 5. *If there exists a function* $c(\mathbf{t}, \mathbf{u}) \in L_2(\mathscr{D} \times \mathscr{D})$, *satisfying the equation*

$$\Delta(\mathbf{t} - \mathbf{s}) = \int_{\mathscr{D}} c(\mathbf{t}, \mathbf{u}) R_1(\mathbf{u} - \mathbf{s})\, d\mathbf{u}, \tag{23}$$

then the measures $\mu_1^{\mathscr{D}}$ *and* $\mu_2^{\mathscr{D}}$ *are equivalent.*

3. Conditions of Absolute Continuity of Measures Corresponding to Homogeneous and Isotropic Random Fields

Let us now consider certain conditions for absolute continuity of measures corresponding to homogeneous isotropic random fields.

We suppose at the outset that the random fields $\xi_1(\mathbf{t})$ and $\xi_2(\mathbf{t})$ are observed on the sphere $S_n(r)$ of radius r in $\mathbf{R}^n$. Let μ_1 and μ_2 be the measures induced by $\xi_1(\mathbf{t})$ and $\xi_2(\mathbf{t})$ in the space of functions $L_2(S_n(r))$. We take as basis in $L_2(S_n(r))$ the spherical harmonics

$$\{S_m^l(\theta_1, \cdots, \theta_{n-2}, \varphi): m = 0, 1, \cdots; l = 1, \cdots, h(m, n)\}.$$

Then it is easy to verify using (5) that the correlation operators $\mathscr{B}_1$ and $\mathscr{B}_2$ of μ_1 and μ_2 are diagonal and the eigenvalues of the operator $D = \mathscr{B}_2^{-1/2}\mathscr{B}_1\mathscr{B}_2^{-1/2} - I$ are equal to

$$\delta_{ml} = \frac{b_m^{(1)}(r)}{b_m^{(2)}(r)} - 1, \qquad m = 0, 1, \cdots; l = 1, \cdots, h(m, n),$$

where

$$b_m^{(i)}(r) = \int_0^\infty \frac{J^2_{m+(n+2)/2}(\lambda r)}{(\lambda r)^{n-2}} d\Phi_i(\lambda). \tag{24}$$

From Assertion A we obtain

Theorem 6. *The measures μ_1 and μ_2 are equivalent if and only if*

$$\sum_{m=0}^\infty h(m, n)\left[\frac{b_m^{(1)}(r)}{b_m^{(2)}(r)} - 1\right]^2 < \infty. \tag{25}$$

In this case,

$$\frac{d\mu_2}{d\mu_1} = \exp\left\{-\frac{1}{2}\sum_{m=0}^\infty \sum_{l=1}^{h(m,n)} \left[\frac{b_m^{(1)}(r) - b_m^{(2)}(r)}{b_m^{(1)}(r)b_m^{(2)}(r)}\right]\right.$$

$$\left.\times \left[\left(\int_0^\infty \frac{J_{m+(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}} Z_m^l(d\lambda)\right)^2 - \log\frac{b_m^{(1)}(r)}{b_m^{(2)}(r)}\right]\right\}. \tag{26}$$

Note that from the spectral representation (5) for $\xi(\mathbf{t})$ it follows that

$$\int_0^\infty \frac{J_{m+(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}} Z_m^l(d\lambda) = \frac{1}{c_n}\int_{S_n} \xi(r, \mathbf{u}) S_m^l(\mathbf{u}) m_n(d\mathbf{u}),$$

where $\mathbf{u} \in S_n(1)$, $m_n(d\mathbf{u})$ is Lebesgue measure on $S_n(1)$.

We suppose additionally that

$$\sum_{m=0}^\infty h(m, n)\left[\frac{b_m^{(1)}(r)}{b_m^{(2)}(r)} - 1\right] \tag{27}$$

converges. The convergence of (25) and (27) ensures that of the infinite product

$$A_n(r) = \prod_{m=0}^{\infty} \left[\frac{b_m^{(1)}(r)}{b_m^{(2)}(r)}\right]^{h(m,n)/2}, \tag{28}$$

and the expression (26) for the density of μ_1 and μ_2 can be written as

$$\frac{d\mu_2}{d\mu_1} = A_n(r) \exp\left\{-\frac{1}{2}\int_{S_n(1)}\int_{S_n(1)} l(\mathbf{u}, \mathbf{v}, r)\xi(r, \mathbf{u})\xi(r, \mathbf{v}) m_n(d\mathbf{u}) m_n(d\mathbf{v})\right\}, \tag{29}$$

where

$$l(\mathbf{u}, \mathbf{v}, r) = \frac{1}{c_n^2} \sum_{m=0}^{\infty} \frac{b_m^{(1)}(r) - b_m^{(2)}(r)}{b_m^{(1)}(r) b_m^{(2)}(r)} \sum_{l=1}^{h(m,n)} S_m^l(\mathbf{u}) S_m^l(\mathbf{v}).$$

By a theorem for adding spherical harmonics,

$$\sum_{l=1}^{h(m,n)} S_m^l(\mathbf{u}) S_m^l(\mathbf{v}) = \frac{h(m, n)}{\omega_n} \frac{C_m^{(n-2)/2}(\cos\langle\mathbf{u}, \mathbf{v}\rangle)}{C_m^{(n-2)/2}(1)},$$

where $\cos\langle\mathbf{u}, \mathbf{v}\rangle$ is the angular distance between $\mathbf{u}$ and $\mathbf{v}$, and ω_n is the area of the surface of the unit sphere in $\mathbf{R}^n$, and the $C_m^\nu(x)$ are the Gegenbauer polynomials defined for $\nu \neq 0$ (i.e., $n \neq 2$) by means of the generating function

$$(1 - 2xt + t^2)^{-\nu} = \sum_{m=0}^{\infty} C_m^\nu(x) t^m.$$

Hence, for $n > 2$,

$$l(\mathbf{u}, \mathbf{v}, r) = \frac{1}{(n-2)(2\pi)^n} \sum_{m=0}^{\infty} \frac{b_m^{(1)}(r) - b_m^{(2)}(r)}{b_m^{(1)}(r) b_m^{(2)}(r)} C_m^{(n-2)/2}(\cos\langle\mathbf{u}, \mathbf{v}\rangle),$$

and for $n = 2$,

$$l(\varphi, \psi, r) = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \frac{b_m^{(1)}(r) - b_m^{(2)}(r)}{b_m^{(1)}(r) b_m^{(2)}(r)} \exp\{-im(\varphi - \psi)\}.$$

Let $\mathscr{D}$ be any bounded region in $\mathbf{R}^n$. We suppose that $\Phi_1(d\lambda)$ and $\Phi_2(d\lambda)$ are absolutely continuous with respect to Lebesgue measure and $\varphi_j(\lambda) = \Phi_j(d\lambda)/d\lambda$. The sufficient condition of equivalence formulated in Theorem 4 takes on in the case of homogeneous and isotropic fields the following form.

Theorem 7. *If*

$$\int_0^\infty \left[\frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{\varphi_1(\lambda)}\right]^2 \lambda^{n-1}\, d\lambda < \infty, \tag{30}$$

then the measures $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$ are equivalent for every bounded region $\mathscr{D}$.

Now let $\mathscr{D}$ be a ball of radius R in $\mathbf{R}^n$ and

$$b_m^{(i)}(r_1, r_2) = \int_0^\infty \frac{J_{m+(n-2)/2}(\lambda r_1)}{(\lambda r_1)^{(n-2)/2}} \frac{J_{m+(n-2)/2}(\lambda r_2)}{(\lambda r_1)^{(n-2)/2}}\, d\Phi_i(\lambda). \tag{31}$$

From (15) and (23), using the representation (5) and the property of orthogonality of spherical harmonics, we obtain the following sufficient conditions for equivalence of $\mu_1^{\mathscr{D}}$ and $\mu_2^{\mathscr{D}}$.

Theorem 8. *If there exists a sequence of functions* $c_m(\rho_1, \rho_2)$ *such that*

1) $c_m(\rho_1, \rho_2)\rho_1^{(n-1)/2}\rho_2^{(n-1)/2} \in L_2([0, R] \times [0, R])$,

2) $b_m^{(2)}(r_1, r_2) - b_m^{(1)}(r_1, r_2)$

$$= c_n \int_0^R \int_0^R c_m(\rho_1, \rho_2) b_m^{(1)}(\rho_1, r_1) b_m^{(1)}(r_2, \rho_2)\rho_1^{n-1}\rho_2^{n-1}\, d\rho_1\, d\rho_2,$$

$$0 \leqq r_1 \leqq R, 0 \leqq r_2 \leqq R,$$

3) $\displaystyle\sum_{m=0}^{\infty} h(m, n) \int_0^R \int_0^R c_m^2(\rho_1, \rho_2)\rho_1^{n-1}\rho_2^{n-1}\, d\rho_1\, d\rho_2 < \infty$,

then the measures $\mu_1^{\mathscr{D}}$ *and* $\mu_2^{\mathscr{D}}$ *are equivalent.*

Note here that the function $c(\mathbf{u}, \mathbf{v})$ in (15) has the form

$$c(\rho_1, \gamma_1, \rho_2, \gamma_2) = \sum_{m=0}^{\infty} h(m, n) c_m(\rho_1, \rho_2) \frac{C_m^{(n-2)/2}(\cos\langle\gamma_1, \gamma_2\rangle)}{C_m^{(n-2)/2}(1)},$$

where (ρ_1, γ_1), (ρ_2, γ_2), $\gamma_1 \in S_n$, $\gamma_2 \in S_n$, are the polar coordinates of $\mathbf{u}$ and $\mathbf{v}$.

Theorem 9. *If there exists a sequence of functions* $\alpha_m(r_1, r_2)$ *such that*

1) $\alpha_m(r_1, r_2) r_1^{(n-1)/2} r_2^{(n-1)/2} \in L_2([0, R] \times [0, R])$,

2) $\displaystyle b_m(r_1, r_2) - b_m(r_1, r_2) = \int_0^R b_m(r_2, \rho)\alpha_m(r_1, \rho)\rho^{n-1}\, d\rho$,

$$0 \leqq r_1 \leqq R, \qquad 0 \leqq r_2 \leqq R,$$

3) $\displaystyle\sum_{m=0}^{\infty} h(m, n) \int_0^R \int_0^R \alpha_m^2(r_1, r_2) r_1^{n-1} r_2^{n-1}\, dr_1\, dr_2 < \infty$,

then the measures $\mu_1^{\mathscr{D}}$ *and* $\mu_2^{\mathscr{D}}$ *are equivalent.*

Received by the editors
August 9, 1971

REFERENCES

[1] Ya. Gaek, *On a property of normal distributions of an arbitrary stochastic process*, Czech. Math. Journal, 8 (83) (1958), pp. 610–618. (In Russian.)

[2] J. Feldman, *Equivalence and perpendicularity of Gaussian processes*, Pacif. J. Math., 8, 4 (1958), pp. 699–708.

[3] Yu. A. Rozanov, *On the density of one Gaussian measure with respect to another*, Theory Prob. Applications, 7, 1 (1962), pp. 82–87.

[4] Yu. A. Rozanov, *On probability measures in functional spaces corresponding to stationary Gaussian processes*, Theory Prob. Applications, 9, 3 (1964), pp. 404–420.

[5] Yu A. Rozanov, *Infinite-dimensional Gaussian distributions*, Trudy Matem. In-ta im. V. A. Steklova, CVIII, 1968. (In Russian.)

[6] I. I. Gikhman and A. V. Skorokhod, *On densities of probability measures in functional spaces*, Uspekhi Matem. Nauk, 21, 6 (1966), pp. 83–152. (In Russian.)

[7] I. I. GIKHMAN and A. V. SKOROKHOD, *Introduction to Theory of Random Processes*, Saunders, Philadelphia, 1969.

[8] M. I. YADRENKO, *On absolute continuity of measures corresponding to Gaussian homogeneous random fields*, Teoriya Veroyat. i Matem. Statistika, Mazhved. Nauch. Sb., vyp. 7 (1972). (In Russian.)

[9] Z. S. ZERAKIDZE, *On the equivalence of distributions of Gaussian homogeneous fields*, Trudy Instituta Prikladnoi Matematiki Tbilisskovo Universiteta, II (1969), pp. 215–220. (In Russian.)

[10] Z. S. ZERAKIDZE, *Some problems on equivalence of distributions of Gaussian homogeneous random fields*, Dissertation, Tbilisi. University, 1971. (In Russian.)

[11] G. M. MOLCHAN and YU. I. GOLOSOV, *Gaussian stationary processes with asymptotic spectrum*, DAN SSSR, 184, 3 (1969), pp. 546–549. (In Russian.)

[12] A. M. YAGLOM, *Second-order homogeneous random fields*, Proc. 4-th Berkeley Sympos. Math. Statist. Prob., II (1961), pp. 593–622.

[13] YU. D. POPOV and M. I. YADRENKO, *Some questions of spectral theory for homogeneous and isotropic random fields*, Theory Prob. Applications, 14, 3 (1969), pp. 510–517.

[14] H. BATEMAN and P. ERDÉYI, *Higher Transcendental Functions*, Volume 2, McGraw-Hill, New York, 1955.

[15] L. I. RONKIN, *Introduction to the Theory of Entire Functions of Several Variables*, Moscow, Izd-vo "Nauka", 1971. (In Russian.)

[16] I. A. IBRAGIMOV and YU. A. ROZANOV, *Gaussian Random Processes*, Izd-vo "Nauka", 1970. (In Russian.)

[17] L. N. SLOBODETSKII, *Generalized S. L. Sobolev spaces and their application to boundary value problems*, Unchenye Zapiski Leningradskovo Pedagogicheskovo Instituta, 197 (1958), pp. 54–112. (In Russian.)

[18] L. R. VOLEVICH and B. P. PANEYAKH, *Certain spaces of generalized functions and embedding theorems*, Uspekhi Matem. Nauk, 20, 1 (1965), pp. 3–74. (In Russian.)

[19] S. M. NIKOL'SKII, *Approximation of functions of several variables and embedding theorems*, Moscow, Izd-vo "Nauka", 1969. (In Russian.)