Solutions for Homework 8

1. (a) Null hypotheses \( H_0: \ p = 0.25 \), alternative hypotheses \( H_A: \ p > 0.25 \). \( p \) is the chance of remission with the treatment.

(b) \( \alpha = P(\text{Type I error}) = \text{P(reject } H_0|H_0) = P(X \geq 150|p = 0.25) \)

Using normal approximation to Binomial data,
\[ \alpha = P(Z > \frac{150 - 500(0.25)}{\sqrt{500(0.25)(0.75)}}) = P(Z > 2.582) = .0049 \]

(c) Power = \( \text{P(reject } H_0|p = 0.35) = P(X \geq 150|p = 0.35) = P(Z > \frac{150 - 500(0.35)}{\sqrt{500(0.35)(0.65)}}) \)
\[ = P(Z > -2.344) = .9905 \]

(d) P-value = \( \text{P}(X \geq 154|p = 0.25) = P(Z > \frac{154 - 500(0.25)}{\sqrt{500(0.25)(0.75)}}) = P(Z > 2.995) = .0014 \)

2. (paired experiment)

The yield differences of the two varieties are: -2.2 -1.3 0.0 -2.0 -1.8 -0.8 1.2 -4.1 -1.6 -0.9

(a) We need to assume that the yield differences of the two varieties is normally distributed. The normality assumption can be checked by a normal score plot:

(b) Test \( H_0: \mu_1 - \mu_2 = 0 \) versus \( H_A: \mu_1 - \mu_2 \neq 0 \).

\( \bar{d} = -1.35, \ s_d = 1.407, \) the test statistic \( t = \frac{-1.35 - 0}{1.407/\sqrt{9}} = -3.034, \)

using a t-distribution with degree of freedom 9, we can get the p-value of the test:
\[ 2(0.005) < \text{p-value} = 2P(T < -3.034) < 2(0.01), \] i.e., \( .01 < \text{p-value} < .02. \)

(c) A 99% C.I. for \( \mu_1 - \mu_2 \) is given by:
\[ \bar{d} \pm t_{0.005} \frac{s_d}{\sqrt{n}} = -1.35 \pm 3.250 \frac{1.407}{\sqrt{10}} = -1.35 \pm 1.446 = (-2.796, 0.096). \]

The interval covers 0, so \( H_0 \) won’t be rejected at \( \alpha = 0.01. \)

3. (independent samples with same sample size)

Species A: \( \bar{x}_1 = 4.764, \ s_1^2 = 0.25, \ n_1 = 9, \)
Species B: \( \bar{x}_2 = 4.242, \ s_2^2 = 0.29, \ n_2 = 9. \)

(a) We assume independence, normality and equal variance.
(b) Test $H_0: \mu_1 - \mu_2 = 0$ versus $H_A: \mu_1 - \mu_2 \neq 0$.

since $n_1 = n_2$, $s^2 = \frac{s_1^2 + s_2^2}{2} = 0.27$, $s(\bar{x}_1 - \bar{x}_2) = \sqrt{2s^2/9} = 0.245$.

Test statistic $t = \frac{\bar{x}_1 - \bar{x}_2}{s(\bar{x}_1 - \bar{x}_2)} = \frac{4.764 - 4.242}{0.245} = 2.13$.

.02 < p-value = $2P(T > 2.13) < .05$ (degrees of freedom of the t-distribution is 16). We have moderate evidence that the two Species don’t have the same mean egg weight.

(c) A 95% C.I. for $\mu_1 - \mu_2$ is given by:

$$\bar{x}_1 - \bar{x}_2 \pm t_{0.025}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = (4.764 - 4.242) \pm 2.120\sqrt{\frac{1}{9} + \frac{1}{9}}$$

$$= .522 \pm .5197 = (.0023, 1.0417)$$

(d) Test $H_0: \mu_1 - \mu_2 = -0.5$ versus $H_A: \mu_1 - \mu_2 \neq -0.5$.

Test statistic $t = \frac{(\bar{x}_1 - \bar{x}_2)-(-0.5)}{s(\bar{x}_1 - \bar{x}_2)} = \frac{4.764 - 4.242 + 0.5}{0.245} = 4.17$.

p-value = $2P(T > 4.17) < .002$. We have strong evidence that the mean egg weight of Species B eggs doesn’t equal the mean weight of Species A eggs plus 0.5.

4. (Independent samples with unequal sample sizes)

Tree 1: $\bar{x}_1 = 145.5$, $s_1^2 = 164.76$, $n_1 = 13$, Tree 2: $\bar{x}_2 = 154.1$, $s_2^2 = 162.77$, $n_2 = 10$,

Assumptions: independent random samples from normal populations with equal variances. We can use Levene’s method to check the assumption of equal variances ($\sigma_1^2 = \sigma_2^2$):

1) The median for Tree 1 is 145, and the median for Tree 2 is 155.5.

2) Calculate the absolute value of all deviations from the median. We get:

Tree 1: 24 25 12 0 2 12 5 9 17 6 2 4 6

Tree 2: 2.5 8.5 18.5 26.5 13.5 12.5 6.5 7.5 3.5 23.5

3) Delete the value 0 in Tree 1.

Tree 1: 24 25 12 2 12 5 9 17 6 2 4 6

Tree 2: 2.5 8.5 18.5 26.5 13.5 12.5 6.5 7.5 3.5 23.5

4) Perform a T-test for comparing the means of the two lists of numbers (with variances assumed equal).

There’s no evidence that the variances $\sigma_1^2$ and $\sigma_2^2$ differ, by calculating P-value.

So, we will perform a T-test assuming equal variances.

Test $H_0: \mu_1 - \mu_2 = 0$ versus $H_A: \mu_1 - \mu_2 \neq 0$.

$$s^2_p = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2} = \frac{(13-1)164.76+(10-1)162.77}{13+10-2} = \frac{3442.05}{21} = 163.91,$$

$$s(\bar{x}_1 - \bar{x}_2) = \sqrt{s^2_p\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 5.38.$$

Test statistic $t = \frac{\bar{x}_1 - \bar{x}_2}{s(\bar{x}_1 - \bar{x}_2)} = \frac{145.5 - 154.1}{5.38} = -1.598$.

p-value = $2P(T > 1.59) < .20$ (degrees of freedom of the t-distribution is 21). We have strong evidence that the mean weights of the samaras of tree 1 and tree 2 differ.

5. Test $H_0: \sigma^2 = 4$ versus $H_A: \sigma^2 > 4$ at $\alpha = 0.05$. $n = 15$.

We will reject $H_0$ if the sample variance $s^2 > a$. 


0.05 = \alpha = P(\text{reject } H_0|H_0) = P(s^2 > a|\sigma^2 = 4) = P(V^2 = \frac{(15-1)s^2}{4} > \frac{(15-1)a}{4}), \text{ where } V^2 \text{ has } \chi^2\text{-distribution with } d.f. = 14.

So, \frac{(15-1)s}{4} = 23.68, a = \frac{4(23.68)}{14-1} = 6.77. \text{ Rejection region: we will reject } H_0 \text{ if } s^2 > 6.77.

Power at \sigma^2 = 13 is

\[ P(\text{reject } H_0|\sigma^2 = 13) = P(s^2 > 6.77|\sigma^2 = 13) = P(V^2 > \frac{(15-1)6.77}{13}) = P(V^2 > 7.29) \approx .925. \]