1. Consider the discrete random variable $X = \text{“number rolled”}$ on a single die with probability mass function 

$$f(x) = \frac{x}{21}, \text{ for } x = 1, 2, 3, 4, 5, 6.$$  

(a) Show that this is a legitimate pmf, by confirming both required conditions. (Hint: Construct a probability chart. You may want to do so in rows rather than columns, to save space.) Show all work. (5 pts)

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\frac{1}{21}$</td>
<td>$\frac{2}{21}$</td>
<td>$\frac{3}{21}$</td>
<td>$\frac{4}{21}$</td>
<td>$\frac{5}{21}$</td>
<td>$\frac{6}{21}$</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>$\frac{1}{21}$</td>
<td>$\frac{3}{21}$</td>
<td>$\frac{6}{21}$</td>
<td>$\frac{10}{21}$</td>
<td>$\frac{15}{21}$</td>
<td>$\frac{21}{21} = 1$</td>
</tr>
</tbody>
</table>

Clearly,

$$f(x) \geq 0, \text{ and } \sum f(x) = \frac{1}{21} + \frac{2}{21} + \frac{3}{21} + \frac{4}{21} + \frac{5}{21} + \frac{6}{21} = 1.$$  

(b) Sketch the probability histogram for this pmf below. (6 pts)

(c) Sketch a graph of the cumulative distribution function $F(x) = P(X \leq x)$ below. (8 pts)
PROBLEM 1 CONTINUED…

(d) Calculate the mean \( \mu \). Show all work! (4 pts)

\[
\mu = E[X] = \sum x f(x) \\
= \frac{91}{21}, \text{ i.e., } 13/3 = 4.333
\]

(e) Set up BUT DO NOT EVALUATE an expression for the variance \( \sigma^2 \). (3 pts)

\[
\sigma^2 = E[(X - \mu)^2] = \sum (x - \mu)^2 f(x) = \\
\left[ (1 - \frac{13}{3})^2 \left( \frac{1}{21} \right) + (2 - \frac{13}{3})^2 \left( \frac{2}{21} \right) + (3 - \frac{13}{3})^2 \left( \frac{3}{21} \right) + (4 - \frac{13}{3})^2 \left( \frac{4}{21} \right) + (5 - \frac{13}{3})^2 \left( \frac{5}{21} \right) + (6 - \frac{13}{3})^2 \left( \frac{6}{21} \right) \right] \\
\text{- OR -} \\
\sigma^2 = E[X^2] - E[X]^2 = \sum x^2 f(x) - \mu^2 = \\
\left[ (1)^2 \left( \frac{1}{21} \right) + (2)^2 \left( \frac{2}{21} \right) + (3)^2 \left( \frac{3}{21} \right) + (4)^2 \left( \frac{4}{21} \right) + (5)^2 \left( \frac{5}{21} \right) + (6)^2 \left( \frac{6}{21} \right) \right] - \left( \frac{13}{3} \right)^2
\]

(f) Suppose someone performs repeated trials of the following game. After the die is tossed, the player wins $2.50 if either 1, 2, or 3 comes up, but loses $1.00 if either 4, 5, or 6 shows up. Over the long run, should the player expect to win, lose, or break even (indicate which), and how much? (Assume the game costs nothing to play.) Show all work. (4 pts)

Let \( Y = \) “$ won (+2.5) or lost (–1) per trial.” Then the (optional) probability chart for \( Y \) would be:

<table>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{1}{21} )</td>
<td>( \frac{2}{21} )</td>
<td>( \frac{3}{21} )</td>
<td>( \frac{4}{21} )</td>
<td>( \frac{5}{21} )</td>
<td>( \frac{6}{21} )</td>
</tr>
<tr>
<td>( y )</td>
<td>+2.5</td>
<td>+2.5</td>
<td>+2.5</td>
<td>–1</td>
<td>–1</td>
<td>–1</td>
</tr>
</tbody>
</table>

Therefore, the expected value of \( Y \) is

\[
\mu_Y = (2.5)(\frac{1}{21}) + (2.5)(\frac{2}{21}) + (2.5)(\frac{3}{21}) + (-1)(\frac{4}{21}) + (-1)(\frac{5}{21}) + (-1)(\frac{6}{21}) = (2.5)(\frac{6}{21}) - (\frac{15}{21}) - (\frac{15}{21}) = 0; \text{ break even}
\]
2. According to the National Institutes of Health, 1 out of 125 babies born in the United States has Congenital Heart Disease (CHD). A random sample of \( n = 775 \) babies are to be selected for a study. Under the assumption of independence of “CHD / No CHD” outcomes between any two babies, answer the following.

(a) Using the Binomial distribution, find the mean number of babies in the sample with CHD, and the variance. Show all work.

\[ \mu = n \pi = (775)(0.008), \text{ i.e., } \mu = 6.2 \text{ babies.} \]
\[ \sigma = \sqrt{n \pi (1 - \pi)} = \sqrt{(775)(0.008)(0.992)}, \text{ i.e., } \sigma = 2.48 \text{ babies.} \]

(b) Continuing with the Binomial distribution, set up BUT DO NOT EVALUATE an expression for the probability that there are exactly 6 babies in the sample with CHD.

\[ P(X = x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x}, \text{ therefore } P(X = 6) = \binom{775}{6}(0.008)^6(0.992)^{769}. \]

(c) Using the Poisson distribution, set up BUT DO NOT EVALUATE an expression for the probability that there are exactly 6 babies in the sample with CHD.

\[ P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}; \text{ therefore, } P(X = 6) = \frac{e^{-6.2} (6.2)^6}{6!}. \]

(d) Numerically evaluate ONE of the expressions in (b) or (c).

\[ P(X = 6) = \frac{e^{-6.2} (6.2)^6}{6!} = 0.1601; \text{ Binomial yields } 0.1607. \]

(e) If each of the babies in the sample of \( n = 775 \) is to be examined one at a time, calculate the probability that the first with CHD will be the 125th. (Hint: Geometric distribution)

\[ P(X = 125) = 0.992^{124}(0.008) = 0.003. \]

(f) Calculate the probability that the first baby with CHD will be the 125th or greater. (Hint: What does each term in the Geometric distribution density function represent?)

\[ P(X \geq 125) \]

\[ P(X = 125) = 0.992^{124}(0.008) = 0.003. \]

\[ P(X \geq 125) = 0.992^{124}(0.008) = 0.3694. \]

Therefore, the first and only Success must occur at the 125th trial or after. (Note that in part (e), this occurs at the 125th trial.)
3. Suppose a continuous random variable $X$ has the **probability density function** shown below.

![Probability Density Function Graph]

(a) Formally show that this is a legitimate pdf, by confirming both required conditions. (3 pts)

> **Solution:** $f(x) \geq 0$ is clear. To show that the total area = 1, it suffices to observe that $f(x)$ forms a right triangle in $0 \leq x < 2$, whose area = $(1/2)(2 - 0)(0.6) = 0.6$, and a rectangle in $2 \leq x \leq 6$, whose area = $(6 - 2)(0.1) = 0.4$. Taking their sum proves the result.

(b) Determine the **cumulative distribution function** $F(x) = P(X \leq x)$, and sketch its graph. (7 pts)

> **Solution:** No calculus required! Clearly, there is no cumulative area if $x < 0$, so $F(x) = 0$ there. The area of the right triangle to the left of any $x$ in $[0, 2)$ however, is equal to $(1/2)(x - 0)(0.3x) = 0.15x^2$. The area to the left of any $x$ in $[2, 6]$ consists of the full right triangle with area 0.6 from (a), plus the rectangular area $(x - 2)(0.1)$, i.e., $0.4 + 0.1x$. This is equal to 1 when $x = 6$, as it should. For $x > 6$, no more cumulative area is added, so $F(x)$ remains at 1 thereafter. Thus, we have

$$F(x) = \begin{cases} 
0, & x < 0 \\
0.15x^2, & 0 \leq x < 2 \\
0.4 + 0.1x, & 2 \leq x \leq 6 \\
1, & x > 6
\end{cases}$$

Note that $F$ rises continuously from 0 to 1.
(c) Calculate the **mean** \( \mu \). **Show all work!**  

**Solution:** By definition,

\[
\mu = E[X] = \int_{-\infty}^{\infty} x f(x) \, dx \\
= \int_{0}^{2} x f(x) \, dx + \int_{2}^{6} x f(x) \, dx \\
= 0.3 \int_{0}^{2} x^2 \, dx + 0.1 \int_{2}^{6} x \, dx \\
= 0.3 \left[ \frac{x^3}{3} \right]_{0}^{2} + 0.1 \left[ \frac{x^2}{2} \right]_{2}^{6} \\
= 0.1 \left[ x^3 \right]_{0}^{2} + 0.05 \left[ x^2 \right]_{2}^{6} \\
= 0.1 (2^3 - 0^3) + 0.05 (6^2 - 2^2) \\
= 0.8 + 1.6 \\
= 2.4
\]

(d) Set up BUT DO NOT EVALUATE an expression for the **variance** \( \sigma^2 \).

**Solution:** By definition,

\[
\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \\
= \int_{0}^{2} (x - 2.4)^2 (0.3x) \, dx + \int_{2}^{6} (x - 2.4)^2 (0.1) \, dx
\]

or alternatively,

\[
\sigma^2 = E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \mu^2 \\
= \int_{0}^{2} x^2 (0.3x) \, dx + \int_{2}^{6} x^2 (0.1) \, dx - (2.4)^2
\]
4. The project manager of a company has a firm deadline of one year to complete a certain task. From experience, she knows the actual amount of time to complete such a task \(X\) is a random variable, that is well-modeled by a Beta distribution, with shape parameters \(p = \frac{3}{2}\) and \(q = \frac{3}{2}\).

(a) Determine the **median** time to completion. Justify your answer.

**Solution:** Since a Beta\((p, q)\) distribution with shape parameters \(p = q\) is symmetric, it follows that its area is divided into two equal halves by the midpoint of the interval \([0, 1]\), i.e., \(\frac{1}{2}\) yr = 6 months.

(b) Determine the **probability density function** (pdf) \(f(x)\) in simplest possible form. **Show all work.**

**Solution:** For a standard Beta\((p, q)\) distribution (i.e., over \([0, 1]\) as here), the pdf is given by

\[
 f(x) = \frac{1}{\text{B}(p,q)} x^{p-1}(1-x)^{q-1}.
\]

Thus, with \(p = q = \frac{3}{2}\), we have

\[
 f(x) = \frac{1}{\text{B}(\frac{3}{2}, \frac{3}{2})} x^\frac{1}{2} (1-x)^\frac{1}{2}.
\]

The denominator \(\text{B}(\frac{3}{2}, \frac{3}{2})\) can be obtained two ways. The most direct way is from the given fact that this incomplete Beta function \(\int_0^x \! t^{p-1}(1-t)^{q-1} \, dt = \frac{1}{4} \left[ \sin^{-1} \left( \sqrt{x} \right) - (1-2x)\sqrt{x-x^2} \right] \), from which it follows that

\[
 \text{B}(\frac{3}{2}, \frac{3}{2}) = \int_0^1 \! t^{p-1}(1-t)^{q-1} \, dt = \frac{1}{4} \left[ \sin^{-1} \left( \sqrt{\frac{1}{2}} \right) - (1-2\cdot\frac{1}{2})\sqrt{\frac{1}{2}-\frac{1}{4}} \right] = \frac{1}{4} \left[ \frac{\pi}{2} \right] = \frac{\pi}{8}.
\]

Alternatively, from the property that \(\text{B}(p,q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}\), we obtain

\[
 \text{B}(\frac{3}{2}, \frac{3}{2}) = \frac{\left( \Gamma(\frac{3}{2}) \right)^2}{\Gamma(3)}.
\]

Furthermore, from \(\Gamma(n+1) = n!\) for \(n = 0, 1, 2, 3, \ldots\), so that the denominator \(\Gamma(3) = 2! = 2\), and

- \(\Gamma(\frac{3}{2}) = \frac{1}{2} \Gamma(\frac{1}{2}) = \frac{1}{2} \sqrt{\pi}\), so that

\[
 \text{B}(\frac{3}{2}, \frac{3}{2}) = \frac{\left( \frac{1}{2} \sqrt{\pi} \right)^2}{2} = \frac{\pi}{8}.
\]

Hence,

\[
 f(x) = \frac{8}{\pi} \sqrt{x(1-x)} \text{ for } 0 \leq x \leq 1,
\]

and 0 elsewhere.

(c) Determine the probability that the project will be completed by three months. **Show all work.**

**Solution:** \(P(X \leq \frac{1}{4}\text{ yr}) = \int_0^{\frac{1}{4}} f(x) \, dx = \frac{8}{\pi} \int_0^{\frac{1}{4}} \sqrt{x(1-x)} \, dx.\) From the given information, it follows that

\[
 \text{this} = \frac{8}{\pi} \cdot \frac{1}{4} \left[ \sin^{-1} \left( \sqrt{x} \right) - (1-2x)\sqrt{x-x^2} \right] \bigg|_0^\frac{1}{4} = \frac{2}{\pi} \left\{ \left[ \sin^{-1} \left( \sqrt{\frac{1}{4}} \right) - (1-2\cdot\frac{1}{4})\sqrt{\frac{1}{4}-\left(\frac{1}{4}\right)^2} \right] - 0 \right\} = \frac{2}{\pi} \left[ \sin^{-1} \left( \frac{1}{2} \right) \right] = \frac{2}{\pi} \left[ \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right] = \boxed{\frac{1}{3} - \frac{\sqrt{3}}{4\pi}} \approx 0.1955.
\]
(d) Determine the probability that completion time will be between three and six months. **Show all work.**

➢ **Solution:**

\[
P\left(\frac{1}{4}\text{ yr} \leq X \leq \frac{1}{2}\text{ yr}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x) \, dx - \int_{0}^{\frac{1}{4}} f(x) \, dx - \int_{\frac{1}{2}}^{1} f(x) \, dx = \frac{0.5}{0.1955} + \frac{0.1955}{0.5} = 0.3045
\]

(e) Determine the probability that the project will be completed before three months, **given** that it will be completed before six months. **Show all work.**

➢ **Solution:**

\[
P(X \leq \frac{1}{4}\text{ yr} \mid X \leq \frac{1}{2}\text{ yr}) = \frac{P(X \leq \frac{1}{4}\text{ yr} \cap X \leq \frac{1}{2}\text{ yr})}{P(X \leq \frac{1}{2}\text{ yr})} = \frac{0.1955}{0.5} = 0.391
\]

(f) Determine the probability that the project will be completed after six months, **given** that it will be completed after three months. **Show all work.**

➢ **Solution:**

\[
P(X \geq \frac{1}{4}\text{ yr} \mid X \geq \frac{1}{2}\text{ yr}) = \frac{P(X \geq \frac{1}{4}\text{ yr} \cap X \geq \frac{1}{2}\text{ yr})}{P(X \geq \frac{1}{2}\text{ yr})} = \frac{1 - 0.5}{1 - 0.1955} = 0.6215
\]

- \[
\int_{0}^{x} \sqrt{1-t} \, dt = \frac{1}{4} \left[ \sin^{-1} \left( \sqrt{x} \right) - (1-2x) \sqrt{x-x^2} \right]
\]

- In a generic right triangle with sides \(x, y,\) and \(r\) as shown below left, \(\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r}.\)

- \(\pi\) radians = 180°; certain trigonometric values can be determined from “special” right triangles, whose sides are in the proportions shown below, middle and right.