The Joyal Approach to Counting Edge-Labeled Trees

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Many authors have offered proofs of Cayley's classic result that that there are n^{n-2} vertex-labeled trees on n vertices, including Prüfer (1918) and Joyal (1981) [2, p. 174]. Oleg Pikhurko [1] gives an "algorithmic" proof in the style of Prüfer that there are n^{n-3} *edge-labeled* trees on n vertices. Here we show that Joyal's "conceptual" approach also yields the edge-labeled result.

Recall Joyal [2, p. 174] uses a canonical correspondence between permutations and lists to identify the functional digraph of $f : [n] \to [n]$ with a marked rooted tree on $[n]$ as follows. The functional digraph has vertex set $[n]$ and an edge from i to j iff $f(i) = j$. The path from any vertex must eventually feed into a cycle and so this graph consists of trees rooted at the vertices of one or more cycles. The cycles form a permutation on a subset of $[n]$, say $(7, 5, 4, 9)(2)(6, 8, 10)$. There is a canonical form for such a permutation: smallest element first in each cycle, cycles arranged in decreasing order of their first elements: $(6, 8, 10)(4, 9, 7, 5)(2)$. The point is that erasing the parentheses now gives a bijection to the set of *lists* on the same elements, because the first elements of the cycles can be recovered as the left-to-right minima of the list. Joyal uses this bijection to "open out" the cycles into a list, an edge joining each list element (except the last) to its successor. Designate the first list element the root and the last list element the mark. Then, ignoring edge directions, we get a marked rooted tree on $[n]$ that represents $f : [n] \to [n]$. Erasing the mark and root—n^2 possibilities—yields Cayley's n^{n-2} formula.

Now Joyal, as just described, identifies a function $f : [n] \to [n]$ whose fixed points include $1, n-1$ and n (there are n^{n-3} such) with a tree on $[n]$ (rooted at n, marked at 1) in which $n-1$ is the first interior vertex on the (unique) path from n to 1. This is because each of $n, n-1$ and 1 forms a one-element cycle and so $n, n-1, 1$ are necessarily the first, second and last element of the associated list. Redirect all edges *away* from the root n and give each the label of its terminal vertex to produce an edge-labeled tree. The root can be recaptured as a terminal vertex of the unique path in the edge-labeled tree with terminal edges 1 and $n-1$, and the n^{n-3} formula follows.

References

[1] Oleg Pikhurko, Generating edge-labeled trees, *Amer. Math. Monthly* **112**, 2005, 919–921.

[2] Martin Aigner, Günter M. Ziegler, *Proofs from* THE BOOK 3rd ed., Springer, 2003.