Sierpinski's Triangle and the Prouhet-Thue-Morse Word

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

November 18, 2006

Abstract

Sierpinski's triangle is a fractal and the Prouhet-Thue-Morse word is sufficiently chaotic to avoid cubes. Here we observe that there is at least a tenuous connection between them: the Sierpinski triangle is evident in Pascal's triangle mod 2 whose inverse, as an infinite lower-triangular matrix, involves the Prouhet-Thue-Morse word.

Pascal's triangle mod 2 (Fig. 1b) is a discrete version of the fractal known as the Sierpinski triangle [1]. Left-justified, it forms an infinite lower-triangular (0,1)-matrix S with 1s on the diagonal (Fig. 1c).

Pascal's triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$

Fig. 1a

Pascal's triangle mod 2

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 0 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ 1 & & 0 & & 0 & & 0 & & 1 \\ & & & & \cdots & & & & \end{array}$$

Fig. 1b

Pascal's triangle mod 2 as an infinite matrix S

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & \\ 1 & 1 & 1 & 1 & 0 & \\ 1 & 0 & 0 & 0 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Fig. 1c

The Prouhet-Thue-Morse word on a 2-letter alphabet $\{a, b\}$ can be formed as follows. Start with a, switch letters and append to get ab. Again switch letters and append to get $abba$. Repeat to get $abbabaab$ and iterate. The result is the infinite Prouhet-Thue-Morse word $t(a, b)$ (t for Thue) that crops up in diverse contexts [2]. Here we observe that the word t is related to the matrix S.

Theorem 1. *S^{-1} is a $(-1, 0, 1)$-matrix. It has the same pattern of zeroes as S and the nonzero entries in each column form the Prouhet-Thue-Morse word $t(1, -1)$.*

This is an immediate corollary of a more general result, Theorem 4 below.

For nonnegative integers i and j, say i is (binary-)free of j if the binary expansion of i has 0s in the positions where j has 1s, equivalently, if the operation of adding i and j in base 2 involves no "carries". By Kummer's well known criterion for the power of a prime dividing a binomial coefficient [3, Ex. 5.36], i is free of $j \Leftrightarrow \binom{i+j}{j}$ is odd. Thus the matrix S has 0s precisely in the positions (i, j) where $i - j$ is not free of j. For nonnegative integer n, let $b(n)$ denote the sum of the binary digits of n. For example, $6_2 = 110$ and $b(6) = 2$. The Prouhet-Thue-Morse word $t(1, -1)$ has the explicit form $\left((-1)^{b(n)}\right)_{n\geq 0}$ [2]. With x an indeterminate, let $S(x)$ denote the infinite lower-triangular matrix defined by

$$S(x)_{ij} = \begin{cases} x^{b(i-j)} & \text{if } i \geq j \geq 0 \text{ and } i - j \text{ is free of } j \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$S(x) = \begin{pmatrix} 1 &&&&&&&& \\ x & 1 &&&&&&& \\ x & 0 & 1 &&&&&& \\ x^2 & x & x & 1 &&&&& \\ x & 0 & 0 & 0 & 1 &&&& \\ x^2 & x & 0 & 0 & x & 1 &&& \\ x^2 & 0 & x & 0 & x & 0 & 1 && \\ x^3 & x^2 & x^2 & x & x^2 & x & x & 1 & \\ \vdots &&&&&&&& \ddots \end{pmatrix}$$

and $S(1) = S$.

Theorem 2. *$S(x)S(y) = S(x+y)$.*

Proof First, as an example, consider $i = 47$, $j = 9$ and k between j and i of the form displayed, where a, b, c are bits (0 or 1) and a prime superscript indicates the complementary bit: $a' = 1 - a$.

integer	binary expansion					
i	1	0	1	1	1	1
j	0	0	1	0	0	1
$i-j$	1	0	0	1	1	0
k	a	0	1	b	c	1
$i-k$	a'	0	0	b'	c'	0
$k-j$	a	0	0	b	c	0

Here, $i-j$ is free of j. Also, k has 1s where j has 1s and 0s where i has 0s. And so $i-k$ is free of k and $k-j$ is free of j and these are the only such k. In the sum $\sum_{k=j}^{i} S(x)_{ik}S(y)_{kj}$ for the (i,j) entry of $S(x)S(y)$, k contributes $x^{a'+b'+c'}y^{a+b+c}$ and the sum over all bits a, b, c is $(x+y)^3$, the (i,j) entry of $S(x+y)$. This works in general, as we now demonstrate.

Suppose $i \geq j$. The (i,j) entry of $S(x)S(y)$ is $\sum_{k=j}^{i} S(x)_{ik}S(y)_{kj}$. For $0 \leq k \leq i$, if $i-k$ is free of k then both must have 0s in the positions where i has 0s. The bits of k in the positions where i has 1s are arbitrary, and then $i-k$ has the complementary bits in these positions. For example, with $i = 1\,0\,1\,1\,1\,1$ (in binary), k must have the binary form $a\,0\,b\,c\,d\,e$ so that $i-k = a'\,0\,b'\,c'\,d'\,e'$. If, further, $k \geq j$ and $k-j$ is free of j then the bits of k are further restricted: they must be 1 in each position where j has a 1. In short, if $i-k$ is free of k and $k-j$ is free of j, then k must have 0s where i has 0s and 1s where j has 1s and is unrestricted where i has a 1 and j has a 0. In particular, the existence of $k \in [j,i]$ with $i-k$ free of k and $k-j$ free of j implies that i must have a 1 in each position where j has a 1 and hence $i-j$ is free of j. So, if $i-j$ is *not* free of j, then $\big(S(x)S(y)\big)_{ij} = \sum_{k=j}^{i} 0 = 0 = S(x+y)_{ij}$. On the other hand, if $i-j$ is free of j, suppose there are $t \geq 0$ positions where i has a 1 and j has a 0 (and so $b(i-j) = t$). As above, the $k \in [j,i]$ for which $i-k$ is free of k and $k-j$ is free of j are unrestricted in these positions and both $i-k$ and $k-j$ have 0s in all other positions. Hence

$$\begin{aligned}\big(S(x)S(y)\big)_{ij} &= \sum_{k=j}^{i} S(x)_{ik}S(y)_{kj} \\ &= \sum_{(i_1,\ldots,i_t)\in\{0,1\}^t} x^{i_1+\ldots+i_t}y^{i'_1+\ldots+i'_t} \\ &= \sum_{m=0}^{t}\binom{t}{m}x^m y^{t-m} \\ &= (x+y)^t \\ &= S(x+y)_{ij}.\end{aligned}$$

Induction yields

Corollary 3. *For q a positive integer, $S(x)^q = S(qx)$.*

Theorem 4. *For rational r, $S^r = S(r)$.*

Proof For $r = p/q$ with p, q positive integers, this follows from

$$S(p/q)^q \underset{\text{Cor. 3}}{=} S(p) \underset{\text{Cor. 3}}{=} S(1)^p = S^p.$$

For negative r, it now suffices to show that $S^{-1} = S(-1)$ and this follows from

$$S(-1)\,S = S(-1)\,S(1) \underset{\text{Thm. 2}}{=} S(0) = I.$$

Added in Proof. Roland Bacher informs me that he has obtained these results more simply by observing that the $2^k \times 2^k$ upper left submatrix of $S(x)$ is the k-fold Kronecker product of $\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)$ [4, 5]. Emmanuel Ferrand treats similar material in an interesting recent paper [6].

References

[1] MathWorld, The Sierpinski Sieve .

[2] J.-P. Allouche and J. O. Shallit, The Ubiquitous Prouhet-Thue-Morse Sequence in C. Ding. T. Helleseth and H. Niederreiter, eds., Sequences and Their Applications: Proceedings of SETA '98, Springer-Verlag, 1999, pp. 1-16.

[3] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics* (2nd edition), Addison-Wesley, 1994.

[4] Roland Bacher and Robin Chapman, Symmetric Pascal matrices modulo p, *European J. Combin.* **25** (2004), 459–473, http://front.math.ucdavis.edu/math.NT/0212144 .

[5] Roland Bacher, La suite de Thue-Morse et la catégorie **Rec**, *Comptes Rendues Acad. des Sci. Paris* Ser I, **342** (2006), 161–164.

[6] Emmanuel Ferrand, An analogue of the Thue-Morse sequence, preprint, 2006, http://www-fourier.ujf-grenoble.fr/~eferrand/ZTM.pdf .