Cesàro's Integral Formula for the Bell Numbers (Corrected)

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October 3, 2005

In 1885, Cesàro [1] gave the remarkable formula

$$N_p = \frac{2}{\pi e}\int_0^{\pi} e^{e^{\cos\theta}\cos(\sin\theta))}\sin(\,e^{\cos\theta}\sin(\sin\theta)\,)\sin p\theta\ d\theta$$

where $(N_p)_{p\geq 1} = (1, 2, 5, 15, 52, 203, \ldots)$ are the modern-day Bell numbers. This formula was reproduced verbatim in the Editorial Comment on a 1941 Monthly problem [2] (the notation N_p for Bell number was still in use then). I have not seen it in recent works and, while it's not very profound, I think it deserves to be better known.

Unfortunately, it contains a typographical error: a factor of $p!$ is omitted. The correct formula, with n in place of p and using B_n for Bell number, is

$$B_n = \frac{2\,n!}{\pi e}\int_0^{\pi} e^{e^{\cos\theta}\cos(\sin\theta))}\sin(\,e^{\cos\theta}\sin(\sin\theta)\,)\sin n\theta\ d\theta \qquad n \geq 1.$$

The integrand is the imaginary part of $e^{e^{e^{i\theta}}}\sin n\theta$, and so an equivalent formula is

$$B_n = \frac{2\,n!}{\pi e}\mathrm{Im}\left(\int_0^{\pi} e^{e^{e^{i\theta}}}\sin n\theta\ d\theta\right). \tag{1}$$

The formula (1) is quite simple to prove modulo a few standard facts about set partitions. Recall that the Stirling partition number $\left\{{n \atop k}\right\}$ is the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into k nonempty blocks and the Bell number $B_n = \sum_{k=1}^{n}\left\{{n \atop k}\right\}$ counts all partitions of $[n]$. Thus $k!\left\{{n \atop k}\right\}$ counts ordered partitions of $[n]$ into k blocks (the $k!$ factor serves to order the blocks) or, equivalently, counts surjective functions f from $[n]$

onto $[k]$ (the jth block is $f^{-1}(j)$). Since the number of unrestricted functions from $[n]$ to $[j]$ is j^n, a classic application of the inclusion-exclusion principle yields

$$k!\left\{ {n \atop k} \right\} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n. \tag{2}$$

The trig identity underlying Cesàro's formula is nothing more than the orthogonality of sines on $[0, \pi]$:

$$\int_0^\pi \sin m\theta \sin n\theta \; d\theta = \begin{cases} \frac{\pi}{2} & \text{if } m = n, \\ 0 & \text{if } m \neq n \end{cases}$$

for m, n nonnegative integers. Using the Taylor expansion $e^x = \sum_{m \geq 0} \frac{x^m}{m!}$ and DeMoivre's formula $e^{i\theta} = \cos\theta + i \sin\theta$, it follows that

$$\operatorname{Im}\left(\int_0^\pi e^{j e^{i\theta}} \sin n\theta \; d\theta \right) = \frac{j^n}{n!} \frac{\pi}{2} \tag{3}$$

for integer $j \geq 0$. Now we show that

$$\operatorname{Im}\left(\int_0^\pi \frac{\left(e^{e^{i\theta}} - 1\right)^k}{k!} \sin n\theta \; d\theta \right) = \frac{1}{n!} \left\{ {n \atop k} \right\} \frac{\pi}{2} \tag{4}$$

for integer $k \geq 0$ (of course, $\left\{ {n \atop k} \right\} = 0$ for $n > k = 0$ and for $k > n$).

Proof The binomial theorem implies the left hand side is

$$\begin{aligned} & \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \operatorname{Im}\left(\int_0^\pi e^{j e^{i\theta}} \sin n\theta \; d\theta \right) \\ \underset{(3)}{=} & \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j^n}{n!} \frac{\pi}{2} \\ \underset{(2)}{=} & \frac{1}{n!} \left\{ {n \atop k} \right\} \frac{\pi}{2} \end{aligned}$$

□

Finally, summing (4) over $k \geq 0$ yields Cesàro's formula (1). The Bell numbers have many other pretty representations, including Dobinski's infinite sum formula [3, p. 210]

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

References

[1] M. E. Cesàro, Sur une équation aux différences mêlées, Nouvelles Annales de Math. (3), **4** (1885), 36–40.

[2] H. W. Becker and D. H. Browne, Problem E461 and solution, *Amer. Math. Monthly* **48** (1941), 701–703.

[3] L. Comtet, *Advanced Combinatorics*, D. Reidel, Boston, 1974.

AMS Classification numbers: 05A19, 05A15.