A Note on Downup Permutations and Increasing 0-1-2 Trees

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
Medical Science Center
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

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Abstract

Downup permutations and increasing 0-1-2 trees are equinumerous: both have exponential generating function $\sec x + \tan x$. Here we give an exposition of a bijection between them, due to Robert Donaghey, and its extension to all permutations, due to Kuznetsov et al. Our explicit description makes invertibility obvious.

Consider a permutation as a list of distinct elements. A *downup* permutation on an ordered set $S = \{s_1 < s_2 < \ldots < s_n\}$ is one of the form $(s_{i_1}\, s_{i_2}\, s_{i_3} \ldots s_{i_n})$ with $s_{i_1} > s_{i_2} < s_{i_3} > s_{i_4} < \ldots$. An updown permutation is similar but with the inequalities reversed; alternating means downup or updown. The unjustly obscure Belgian combinatorialist Désiré André found in 1879, with a_n denoting the number (A000111) of downup permutations on an n-element set, that the exponential generating function $A(x) = \sum_{n\geq 0} a_n x^n/n!$ satisfies the differential equation $2A'(x) = A(x)^2 + 1$, and he deduced the famous result [1] that $A(x) = \sec x + \tan x$.

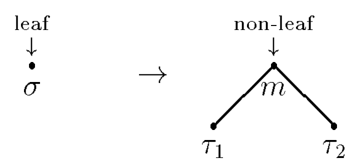
A *0-1-2 tree* is an unordered rooted tree in which the outdegree (= number of children) of each vertex is 0, 1 or 2. (We reserve the term binary tree to connote, as usual, a left/right orientation for each edge.) A vertex of outdegree 0 is a *leaf*, otherwise it is *interior*. An *increasing* 0-1-2 tree on $[n]$ is a 0-1-2 tree with vertices labeled $1, 2, \ldots, n$ in which each child's label is greater than its parent's label. In particular, the root is necessarily labeled 1. One cannot *draw* such a tree without first deciding, for each vertex of outdegree 2, which of its two subtrees goes left and which right. So, for definiteness, put the subtree with the bigger *maximum vertex* on the left and let's designate the edge from a vertex of outdegree 1 to be a *left* edge; now every edge has a left/right orientation and let's call the result a *Donaghey* tree. So a Donaghey tree is simply a binary tree with increasing labels and the *left-largest* property: the largest descendant of each interior vertex is in the left subtree. (We have here an instance of the slightly paradoxical equivalence "unordered unrestricted $\sim$ ordered restricted". The resolution of the paradox is perhaps that "unordered" is a more sophisticated concept than "ordered".)

Clearly, the number of downup permutations on $[n]$ is half the number of alternating permutations on $[n]$ for $n \geq 2$. But for present purposes it is more convenient to bisect the set of alternating permutations in a different way: a *maxmin* permutation is one in which the maximum precedes the minimum and analogously for *minmax*. Thus, for example, 3 1 2 is maxmin but 2 1 3 is not, and half of the alternating permutations on $[n]$ are maxmin when $n \geq 2$. Obviously then, downup permutations on $[n]$ are equinumerous with maxmin alternating permutations on $[n]$ (and the reader might like to supply a bijection between them).

The *reverse* of a permutation $s = (s_1, s_2, \ldots, s_n)$ is $(s_n, s_{n-1}, \ldots, s_1)$ and the *complement* is obtained by interchanging the largest and smallest elements, the second-largest and second-smallest elements, and so on (in case n is odd the median entry is undisturbed). In particular, if s is a permutation on $[n]$, then its complement is $n + 1 - s$ (elementwise). The key properties of the complement operation that we will use below are as follows. For all permutations of length ≥ 2, (i) it preserves the alternating property, (ii) it reverses the rise/fall status of the first 2 entries and also of the last 2 entries, and (iii) it interchanges maxmin and minmax permutations.

Donaghey's bijection is from maxmin alternating permutations on $[n]$ to Donaghey trees on $[n]$. To present it, it is helpful to consider a hybrid notion intermediate between the two: a binary tree such that (i) each interior vertex is labeled with a singleton subset of $[n]$ and each leaf is labeled with a permutation on a subset of $[n]$ and these subsets together form a partition of $[n]$, (ii) for each interior vertex whose (1 or 2) children are leaves, the concatenation of labels (left leaf)·(parent)·(right leaf) is alternating (here an empty leaf label is permissible), (iii) for each interior vertex its label—a single integer—is less than all integers in the labels of its children, and (iv) the largest integer among the labels of the descendants of an interior vertex occurs in the left subtree (in particular, every interior vertex has a nonempty left subtree). Such a hybrid with all labels singletons is simply a Donaghey tree.

Here is the bijection. Given a maxmin alternating permutation, write it as $\pi_1\, 1\, \pi_2$ (one of π_1, π_2 may be the empty permutation). Start with a tree rooted at 1, a left child labeled π_1 and a right child labeled π_2 (except no child if the corresponding permutation is empty). Successively reduce label sizes and grow the hybrid tree as follows. For a leaf with label σ of length ≥ 2, let m be the smallest entry in σ. Turn this leaf into a non-leaf by adding children as shown

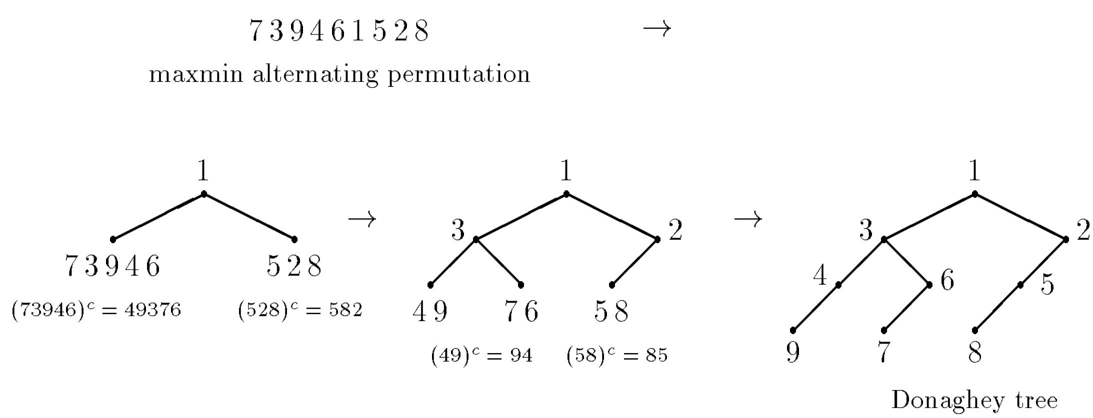


where the concatenation of labels

$$\tau_1 m \tau_2 = \begin{cases} \sigma & \text{if } \sigma \text{ is maxmin;} \\ \sigma^c & \text{otherwise.} \end{cases}$$

(One of τ_1, τ_2 may be empty, in which case that child is absent.)

Stop when all leaves have singleton labels. This is the Donaghey tree corresponding to the given maxmin alternating permutation. Every tree obtained enroute is of the above hybrid type. An example follows.



Bijection from maxmin alternating permutation to Donaghey tree

The reverse construction successively melds a leaf vertex and sibling (if present) into a longer label on the parent. The only issue is determining whether a $\tau_1 m \tau_2$ is σ or σ^c and this is settled by the following fact.

Lemma (i) If m is a left child, then $\tau_1 m \tau_2$ (as defined from σ above) ends with a rise if σ is maxmin and with a fall if σ is minmax, and (ii) if m is a right child, then $\tau_1 m \tau_2$ starts with a fall if σ is maxmin and with a rise if σ is minmax.

Proof (i) Say m is a left child of c. Since σc is alternating and c is its smallest entry, σ ends with a rise. Hence σ maxmin implies $\tau_1 m \tau_2 = \sigma$ ends with a rise and σ minmax implies $\tau_1 m \tau_2 = \sigma^c$ ends with a fall. Case (ii) is analogous. □

This proves the mapping is invertible. Roughly speaking, the construction trades the property of the original permutation of being alternating at an interior entry, when necessary, to ensure the corresponding tree has the left-largest property.

Now, listing the vertices in inorder (a.k.a. symmetric order) [3] is a well known bijection from increasing binary trees on $[n]$ to permutations on $[n]$. So it is natural to ask if Donaghey's bijection can be extended to these sets. It can, and it's quite easy [4]. First extend to all maxmin permutations: perform the same construction as above but, as each interior vertex m is introduced, attach an additional label F (for flip) if the original permutation is *not* alternating at m. When done, successively flip (left $\leftrightarrow$ right) the subtrees of vertices with an F label. If a flip occurs at m, the largest of its descendants will be in the right subtree and so all F labels can be recovered, ensuring invertibility. This map sends maxmin permutations on $[n]$ to those increasing binary trees on $[n]$ for which n occurs in the left subtree of the root. Finally, send a minmax permutation to the flip of the image of its reverse to get the full bijection.

This bijection shows that there are $n!/2$ F-labeled Donaghey trees. Let us count them by number k of leaves. An F may be assigned (or not) to each non-root interior vertex. The number of such vertices is $n-1-k$, giving 2^{n-1-k} labeling choices. So, if $a_{n,k}$ is the number of Donaghey trees with n vertices and k leaves, and a_n is the downup number (A000111), we have

$$\begin{aligned} \sum_{k\geq 1} a_{n,k} &= a_n, \\ \sum_{k\geq 1} 2^{n-1-k} a_{n,k} &= n!/2. \end{aligned}$$

The $a_{n,k}$ are generated by the bilinear recurrence [4]

$$\begin{aligned} a_{n,1} &= 1 & n &\geq 2 \\ a_{n,k} &= k a_{n-1,k} + (n+2-2k) a_{n-1,k-1} & 2 \leq k &\leq \left\lfloor \tfrac{n+1}{2} \right\rfloor \\ a_{n,k} &= 0 & k &> \left\lfloor \tfrac{n+1}{2} \right\rfloor \end{aligned}$$

and the first few values are given in the following table.

$n\backslash k$	1	2	3	4	5
2	1				
3	1	1			
4	1	4			
5	1	11	4		
6	1	26	34		
7	1	57	180	34	
8	1	120	768	496	
9	1	247	2904	4288	496

Donaghey trees by number of leaves

References

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[3] D. E. Knuth, *The Art of Computer Programming*, Vol. 4, pre-fascicle 4a available at http://www-cs-staff.Stanford.EDU/~knuth/news.html

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