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A One-Line Description of the Calkin-Wilf Enumeration of the Rationals

$$\underbrace{\frac{355}{113}}_{\substack{\text{positive}\\ \text{rational}}} = \underbrace{3+\cfrac{1}{7+\cfrac{1}{16}}}_{\substack{\text{continued fraction}\\ \text{of } \textit{odd} \text{ length}}} = \underbrace{[3;7,16]}_{\substack{\text{partial}\\ \text{quotients}}} \longrightarrow \underbrace{\overbrace{11\ldots1}^{16}\overbrace{00\ldots0}^{7}\overbrace{111}^{3}{}_{2}}_{\substack{\text{alternating runs of 1s and 0s}\\ \text{give a binary expansion}}} = \underbrace{67107847}_{\substack{\text{positive}\\ \text{integer}}}$$

Every positive rational has a unique continued fraction expansion of odd length obtained by replacing the last partial quotient a_n by two partial quotient $a_n - 1$ and 1 if necessary. So, for another example, $1/7 = [0;7] = [0;6,1] \rightarrow 1000000_2 = 64$ and the 64th rational in the Calkin-Wilf enumeration is $1/7$.

References

[1] Neil Calkin and Herbert Wilf, Recounting the Rationals, *Amer. Math. Monthly* (**107**) 2000, 360-364.

[2] David M. Bradley, Counting the positive rationals: a brief survey, http://front.math.ucdavis.edu/math.HO/0509025, preprint.