Monthly Problem **10978** by Jean-Pierre Grivaux. *Let* $P_n(x) = x^{n^2+n-1} - 3x^{n^2} + x^{n^2-1} + x^{n^2-n} + x^{2n-1} + x^n - 3x^{n-1} + 1$. *Show that for every integer* $n \geq 2$ *and real* $x \geq 0$, $P_n(x) \geq 0$.

SOLUTION

by David Callan

A *path* is a lattice path of upsteps $(1,1)$ and downsteps $(1,-1)$ that starts at the origin (a list of 1s and -1s for short). A path is *balanced* if it has the same number of upsteps and downsteps, and *nonnegative* if it never dips below the x-axis. A *Dyck path* is a balanced nonnegative path. Encode a path P as a polynomial $P(x) = \sum_i \epsilon_i x^{i-1}$ with $\epsilon_i = 1$ if the ith step of P is up, -1 if it is down. Thus a path P is balanced $\Leftrightarrow P(1) = 0$.

Lemma. Let P be a balanced path.

(i) The coefficients of the polynomial $\frac{P(x)}{1-x}$ are all nonnegative $\Leftrightarrow P$ is a Dyck path.

(ii) If P is a Dyck path consisting of the lattice points $\{P_i\}_{i=0}^{2n}$ (P_0 is the origin), then $(1-x)P(x) = \sum_{i=0}^{2n} a_i x^i$ with $a_0 = 1$, $a_{2n} = -1$ and for $0 < i < 2n$,

$$a_i = \begin{cases} -2 & \text{if } P_i \text{ is a peak of } P \\ 2 & \text{if } P_i \text{ is a trough of } P \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) follows from the observation that the ordinates of the interior lattice points of P are the coefficients of $\frac{P(x)}{1-x}$. (ii) follows from the fact that the a_i $(0 < i < 2n)$ are given by applying the difference operator to the ± 1 sequence defining P.

A Dyck path can also be specified by the lengths of its successive runs of upsteps and downsteps, that is, by its ascents and descents. Now consider the $n(n-1)$-step Dyck path D with ascents $n-1, n-2, \ldots, 1$ and descents $1, 2, \ldots, n-1$ (in that order). Its peaks have abscissae $\big(i(n-1)\big)_{1 \leq i \leq n-1}$ and its troughs have abscissae $\big(j(n+1)\big)_{1 \leq j \leq n-2}$. Part (ii) of the Lemma now permits evaluation of $(1-x)D(x)$ using geometric sums as

$$1 + 2x^n \frac{1 - x^{n(n-1)}}{1 - x^n} - 2x^{n-1} \frac{1 - x^{(n-1)^2}}{1 - x^{n-1}} - x^{n(n-1)},$$

and $(1-x^n)(1-x^{n-1})(1-x)D(x)$ simplifies to the proposed polynomial $P_n(x)$. The former is the product of three polynomials: $(1-x^n)(1-x^{n-1})$, $(1-x)^2$ and $\frac{D(x)}{1-x}$, each of which is ≥ 0 for $x \geq 0$ (the last by part (i) of the Lemma). The proposed assertion follows.